First-Price Auctions with Maxmin Expected Utility Bidders*

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Abstract

This paper studies the first price auction with independent private valuations, wherein each bidder faces ambiguity about the probability distribution from which the other bidders' valuations for the item are drawn. Each bidder is ambiguity averse and this ambiguity is represented by a set of priors. In this informational setting, a maxmin Bayesian Nash equilibrium of the auction is identified. It is also shown that the bidders' bids and the seller's expected revenue increase as the level of the bidders' ambiguity increases if the bidders' valuation distribution satisfies the monotone inverse hazard rate condition. Finally, the paper shows that the seller's expected revenue from the first price auction is greater than that from the second price auction.

KEYWORDS: Auctions, Ambiguity, Maxmin
JEL CLASSIFICATION: D44, D81, D82.

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1 Introduction

Much of the existing literature on auctions with independent private valuations makes the following assumption: Each bidder's valuation for the item is distributed from a unique prior F and this distribution is common knowledge among the bidders in the auction. This paper relaxes this assumption and studies auctions. There are two reasons that the unique prior assumption is weakened in this paper. First, there are real-world examples in which this assumption seems strong. House auctions, merger and acquisition auctions, art auctions, and online auctions are some of these examples. Since the bidders in these auctions rarely learn the auction environment due to the lack of repeated participation, they don't have enough information to have a unique prior. The unique prior assumption is accordingly not suitable for these type of auctions. Second, robustness of the results under the unique prior assumption can be examined by weakening the assumption. Many well-known results from existing literature such as the revenue equivalence theorem or the equilibria of a certain auction format depend crucially on the unique prior assumption. Since Wilson (1989) emphasized the importance of studying mechanisms in a wide class of environments, robustness has been one of the key questions in the study of mechanisms, including auctions. Because of these two reasons, the unique prior assumption is relaxed in this paper and it is assumed that the bidders in an auction face ambiguity about the probability distribution from which each bidder's valuation is drawn and that they are ambiguity averse. The aim of this paper is to study the first-price and the second-price auctions under this relaxed assumption. The paper especially focuses on the first-price auctions because the outcomes of the second-price auctions are trivial.

One of the earliest studies on ambiguity was conducted by Knight (1921). In his book, Knight differentiates ambiguity from risk. Risk refers to situations where probabilities are known; ambiguity, on the other hand, describes situations where probabilities are not known.[1] Ellsberg (1961) subsequently shows in his paper that if an agent faces ambiguity about a state of nature and is ambiguity averse, then the decision-making behavior of the agent cannot be explained by a unique belief. A decision-making rule of ambiguity averse agents is axiomatized by Gilboa and Schmeidler (1989). They introduce the maxmin expected utility model with multiple priors in which an agent has a set of multiple priors, instead of a unique prior, about a state of nature. In this model, the ambiguity of the agent is captured by the set of priors, and the agent's utility from choosing an action is its minimum expected utility across all beliefs in her prior set. She then selects the action that maximizes this minimum expected utility. The model developed by Gilboa and Schmeidler (1989) is adopted in this paper to explain the bidding behavior of bidders who face ambiguity. In this paper, it is assumed that the bidders in the auction do not know the probability distribution from which the other bidders' valuations are drawn; that is, they face ambiguity about the valuation distribution. It is also assumed that they are ambiguity averse. According to the maxmin expected utility model with multiple priors, the ambiguity of each bidder in the auction is represented by a set of multiple beliefs. Each bidder evaluates a bid based on its minimum expected utility across her priors and chooses the best bid.

[1] Knight (1921) uses the term ”uncertainty” instead of ”ambiguity” in his book.

Some researchers have investigated auctions with ambiguity averse bidders by employing the maxmin expected utility decision rule; these researchers include Bose, Ozdenore, and Pape (2006); Bodoh-Creed (2012); and Lo (1998). One of the main differences between my paper and these papers is that my paper focuses on studying how the outcome of the auction changes as the level of ambiguity faced by the bidders changes. My paper quantifies the ambiguity level of each bidder from her prior set. Then, it analyzes how the bidders' ambiguity level affects their bids and the seller's expected revenue from the auction. This is an interesting question for the seller of the auction. If the seller knows how his expected revenue is impacted by the bidders' ambiguity level, then he can increase his expected revenue by adjusting the bidders' information level.

The following are the main results of the paper. A maxmin Bayesian Nash equilibrium of the first price auction is identified. Then, it is shown that the bidders' equilibrium bids and the seller's expected revenue from the auction increase as the bidders' ambiguity level increases if the distribution of the bidders' valuations satisfies the monotone inverse hazard rate condition. Moreover, the paper shows that the first price auction generates a larger expected revenue for the seller than the second price auction and that the revenue gap between the two auction formats increases as the level of ambiguity faced by the bidders increases.

To derive the equilibrium of the auction (Proposition 1), I use the maxmin Bayesian Nash equilibrium as the solution concept. Since this is Nash equilibrium, each bidder's bidding strategy is the best response against the other bidders' strategies according to the maxmin expected utility decision rule. Prior studies that have analyzed games with agents facing ambiguity by using the maxmin Bayesian Nash equilibrium solution concept include Bose, Ozdenoren and Pape (2006), Bodoh-Creed (2012), and Wolitzky (2016).

This paper defines the prior set of each bidder as follows. It is assumed that there is a probability distribution from which each bidder's valuation is independently drawn and that each bidder's prior set is a set of probability distributions in the neighborhood of this true valuation distribution. The Levy metric, an intuitive metric on the set of probability distributions that measures the maximum distance between the graphs of two cumulative distribution functions, is used to define the neighborhood. Then, the level of ambiguity that each bidder faces is represented by the size of the neighborhood. Because the ambiguity level is captured by a parameter in this prior set definition, it is convenient to analyze how the bidders' ambiguity level affects the outcome of the auction. Among the probability distributions in a bidder's prior set defined by the Levy metric, we can consider two distributions: the lower bound distribution and the upper bound distribution. In the prior set, there is a distribution that first-order stochastically dominates all other distributions in the set. I use the term "the lower bound distribution" to denote this distribution because its cumulative distribution function values are lower than those of any other distributions in the set. If a bidder's belief about another bidder's valuation is the lower bound distribution, then compared to the other beliefs in her prior set, she believes that the other bidders' valuations for the item are higher. In the prior set, there is also a distribution that is first-order stocahstically dominated by all other distributions in the set. I use the term "the upper bound distribution" for this distribution. Any probability distribution whose cumulative distribution function values are between those of the lower bound and upper bound distributions are contained in the bidder's prior set.

Consider a maxmin expected utility problem faced by a bidder. Regardless of the bidder's bid, her worst belief, the expected payoff minimizing belief, in her prior set is the probability distribution that first-order stochastically dominates all the other distributions. If the bidder has this belief, then she believes that the other bidders' valuations for the item are high and there is a small chance of her winning the item. Thus, this is the bidder's worst belief. Because the worst belief does not depend on the bid the bidder chooses, the maxmin Bayesian Nash equilibrium with multiple priors is equal to the Bayesian Nash equilibrium based on the worst belief. Thus, the maxmin Bayesian Nash equilibrium of the first-price auction is obtained by using the results of previous research on the Bayesian Nash equilibrium in the first-price auction conducted by Riley and Samuelson (1981) and Monteiro (2009). Riley and Samuelson (1981) derive the equilibrium of the auction when the bidders' valuations are drawn from a continuous distribution, and Monteiro (2009) generalizes this result to the case of distributions with discontinuities.

This paper analyzes how the bidders' bidding behavior and the seller's expected revenue change as the level of ambiguity faced by the bidders changes (Proposition 2, 3). Under the assumption that the true valuation distribution satisfies the monotone inverse hazard rate condition, each bidder in the auction submits a higher bid in response to an increased level of ambiguity. Consider a bidder facing ambiguity. If she faces a higher level of ambiguity, then her worst belief is more pessimistic than the worst belief from the lower level of ambiguity. That is, if the bidder's ambiguity level increases, then she believes that the other bidders' valuations for the item are higher. Thus, to compete against the other bidders with higher valuations, she submits a higher bid. Due to the higher bids of the bidders, the seller's expected revenue is also higher. It follows that the seller's expected revenue from the auction increases as the level of bidders' ambiguity about the distribution of the other bidders' valuations increases.

The first price auction can be compared to the second price auction when bidders have ambiguity (Proposition 4). If bidders do not face ambiguity, there is a well-known result that the first price auction and the second price auction generate the same expected revenue for the seller. However, the first price auction generates greater expected revenue than the second price auction if there is ambiguity. Moreover, the difference in revenues between these two auction formats becomes larger as the bidders' ambiguity level rises. Consider a second price auction. If the bidders don't face ambiguity, it is a dominant strategy for them to bid their own valuations. It is still a dominant strategy even when the bidders face ambiguity because dominant strategies don't depend on the priors that agents have. Therefore, the seller's expected revenue from the second price auction does not depend on whether the bidders have ambiguity or not. As we noted, however, the seller's expected revenue from the first price auction increases with the increases in bidders' ambiguity level. That is, the sensitivity of the auction format to ambiguity is different for the first price and second price auctions, and this leads to the revenue gap between these two auction formats.

The paper is organized as follows. Section 2 discusses related literature. The model and informational assumptions are introduced and each bidder's prior set is defined in section 3. In section 4, a maxmin Bayesian Nash equilibrium of the first price auction is identified. Changes in bidders' bidding behavior and the seller's expected revenue in relation to changes in the bidders' ambiguity level are analyzed in section 5. Section 6 compares the results from the first price auction with those of the second price auction. Section 7 and 8 conclude the paper by offering future research directions.

2 Related literature

There is an existing literature that studies auctions where the bidders face ambiguity about the probability distribution from which the valuations of the other bidders are drawn and they are ambiguity averse. Lo (1998) examines first price and second price sealed-bid auctions using the maxmin expected utility model, showing that the revenue for the seller is greater from the first price auctions than the second price auctions. One of the main differences between my paper and Lo (1998) is that my paper examines how the bidders' ambiguity level affects the outcome of the auction. This is possible because each bidder's ambiguity level can be defined by a parameter determining the bidder's prior set in my paper. Bose, Ozdenore, and Pape (2006) and Bodoh-Creed (2012) study the optimal auction problem and characterize the revenue maximizing auction. My work focuses on one auction format, the first price auction. The first price auctions are not in their set of optimal auctions.

However, they mention that their optimal auctions are rarely observed in the real world unlike the first price auctions.

Other papers have adopted the maxmin expected utility model to explain the decision-making behavior of agents facing ambiguity. Bergemann and Schlag (2011) investigate the monopoly pricing problem when the monopolist has ambiguity about the demand distribution. I adopt their definition of a prior set to define the bidders' prior sets. I use the Levy metric to define the neighborhood of the true distribution, which is how Bergemann and Schlag (2011) define the monopolist's prior set. However, my paper analyzes the bidders' optimal bidding problems in the auction while Bergemann and Schlag (2011) analyze the monopolist's optimal pricing problem. Wolitzky (2016) studies properties of mechanisms using the maxmin expected utility model. He works on mechanisms in general, however, my work focuses on auctions.

Riley and Samuelson (1981) and Monteiro (2009) study the Bayesian Nash equilibrium of the first price auction when the bidders don't have ambiguity. I use their results to derive a maxmin Bayesian Nash equilibrium of the auction.

3 Model

2.1. Auction

There is an indivisible item to be auctioned. Suppose that there are n risk-neutral bidders and the set of the bidders is defined as $N = \{1, 2, \ldots, n\}$. For each bidder $i \in N$, let

$v_i \in V_i \subseteq \mathbb{R}_+$ denote her valuation for the item where V_i is the set of bidder i's possible valuation for the item, $b_i \in B_i$ denote her bid where B_i is the set of bidder i's possible bids, and $b_{-i} \equiv (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$ denote a profile of the bids with bidder i removed.

Consider a first price sealed-bid auction with seller's reserve price r. In the auction, a bidder with the highest bid wins the item and pays her bid, the highest bid, as long as her bid is higher than or equal to the reserve price r. Assume that each highest bidder wins the item with the same probability in case of a tie, $V_i = [0, 1]$, and $B_i = [0, 1]$ for each bidder $i \in N$. Let $p_i(b_1, \ldots, b_n)$ denote the probability that bidder i wins the item and $t_i(b_1, \ldots, b_n)$ denote bidder i's expected payment to the seller when $(b_1, \ldots, b_n)$ is a profile of bids submitted by the bidders. Then, the allocation rule and the transfer rule of the auction are defined as follows: for each $i \in N$ and for each bid profile $b = (b_i, b_{-i}) \in [0, 1]^n$,

$$p_i(b_i, b_{-i}) = \begin{cases} 1 & \text{if } b_i > b_{-i}^{max} \text{ and } b_i \geq r, \\ \dfrac{1}{k} & \text{if } b_i = b_{-i}^{max} \text{ and } b_i \geq r, \\ 0 & \text{if } b_i < b_{-i}^{max} \text{ or } b_i < r, \end{cases} \qquad t_i(b_i, b_{-i}) = \begin{cases} b_i & \text{if } b_i > b_{-i}^{max} \text{ and } b_i \geq r, \\ \dfrac{b_i}{k} & \text{if } b_i = b_{-i}^{max} \text{ and } b_i \geq r, \\ 0 & \text{if } b_i < b_{-i}^{max} \text{ or } b_i < r. \end{cases}$$

where $b_{-i}^{max} \equiv \max_{j \neq i} b_j$ and $k \equiv |\{l \in N \,|\, b_l = b_i\}|$. Thus, the payoff of bidder i with valuation v_i is

$$\begin{aligned} u_i(b_i, b_{-i}; v_i) &= v_i \, p_i(b_i, b_{-i}) - t_i(b_i, b_{-i}) \\ &= \begin{cases} v_i - b_i & \text{if } b_i > b_{-i}^{max} \text{ and } b_i \geq r, \\ \dfrac{v_i - b_i}{k} & \text{if } b_i = b_{-i}^{max} \text{ and } b_i \geq r, \\ 0 & \text{if } b_i < b_{-i}^{max} \text{ or } b_i < r. \end{cases} \end{aligned}$$

From this point, it is assumed that the auction in this paper is the first price auction with reserve price r, $V_i = [0, 1]$ and $B_i = [0, 1]$ for all $i = 1, 2, \ldots, n$ unless stated otherwise.

2.2. Information

Bidder i's valuation for the item, $v_i \in [0, 1]$, is her private information and unknown to the other bidders and the seller. Assume that v_i for each bidder i is independently drawn from the continuously differentiable and strictly increasing distribution F_0 on [0,1] whose density function is f_0. Suppose that the bidders don't know the distribution, that is, they face ambiguity about the distribution and also that they are ambiguity averse . Each bidder knows that the valuations of the other bidders are identically and independently distributed from a distribution but she doesn't know the distribution. The ambiguity of each bidder can be represented by a set of probability distributions. We assume that each bidder's set of beliefs about another bidder's valuation is the set of all probability distributions on [0,1] in ϵ-neighborhood of the distribution F_0. Following Bergemann and Schlag (2011), the Levy metric on the set of probability distributions is used to define the ϵ-neighborhood of F_0.[2] Then, bidder i's set of beliefs is given by

$$\mathcal{F}_\epsilon(F_0) = \{F \in \Delta[0, 1] \,|\, F_0(v - \epsilon) - \epsilon \leq F(v) \leq F_0(v + \epsilon) + \epsilon \ \ \forall v \in [0, 1]\}.$$

The Levy metric measures the maximum distance between the graphs of two cumulative distribution functions along a 45° direction. In the belief set defined by the Levy metric, there are two probability distributions on $[0, 1]$, $F_0(v - \epsilon) - \epsilon$ and $F_0(v + \epsilon) + \epsilon$, that form

[2]In their paper on monopoly pricing, Bergemann and Schlag (2011) use a generalized version of Levy metric to define the seller's set of beliefs about the demand function. See Huber and Ronchetti (2009) on robust statistics for the definition of the Levy metric.

a boundary of the set. The graph of each of these distributions is a parallel shift of the graph of the value distribution, F_0, along a 45° direction. Then, any distribution on $[0, 1]$ whose graph is located between the graphs of these two distributions is in the bidder's belief set. Figure 1 depicts each bidder's set of beliefs when F_0 follows a uniform distribution on $[0, 1]$. Any distribution on $[0, 1]$ whose graph falls in the shaded area in the figure is included in the bidder's belief set. In our definition of the belief set using the Levy metric, the size of the neighborhood, ϵ, represents the level of ambiguity that the bidder has. A higher value of ϵ means the higher level of ambiguity of the bidder because the set of beliefs is larger. Because each bidder's ambiguity level is captured by a parameter, it is convenient to analyze how the bidders' ambiguity level affects the outcome of the auction. It is assumed that the auction rule, the reserve price r, and each bidder's set of beliefs $\mathcal{F}_\epsilon(F_0)$ are common knowledge.

2.3. Maxmin expected utility bidders

To analyze the behavior of bidders facing ambiguity, I adopt the maxmin expected utility decision rule that is axiomatized by Gilboa and Schmeidler (1989). Under the maxmin expected utility decision rule, each bidder calculates the minimum expected payoff across all beliefs for each of her possible bids. Then, she chooses the bid that maximizes the minimum expected payoff. The mathematical formulation of the bidder's minimum expected payoff maximization problem is provided in the next subsection.

2.4. The game-theoretic auction and maxmin Bayesian Nash equilibrium

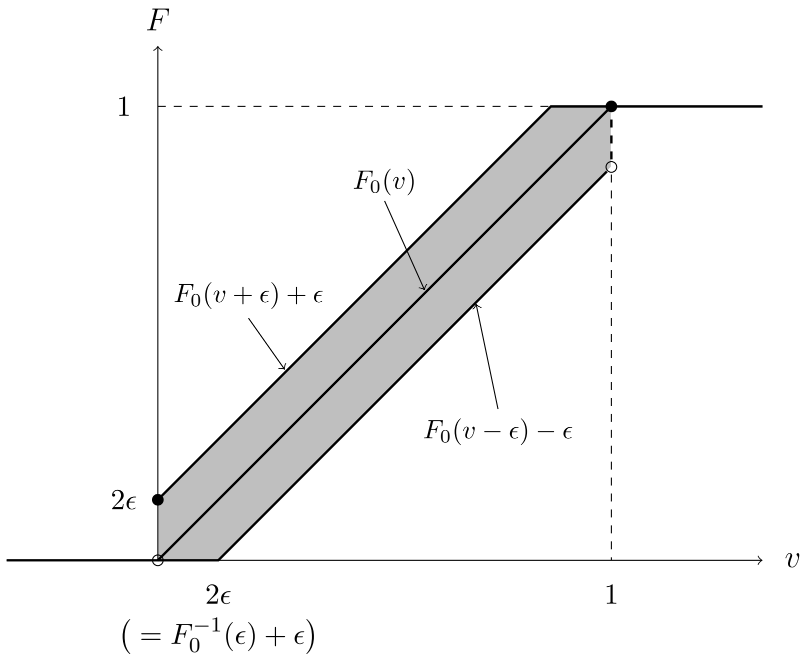

Figure 1: A bidder's set of beliefs when F_0 is a uniform distribution on $[0, 1]$.

Consider the auction as a game. Each bidder's strategy is a bidding function $b_i : V_i \to B_i$. Let $v_{-i} \equiv (v_j)_{j\neq i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ denote a vector of bidders' valuations with bidder i removed. Consider bidder i with the set of beliefs $\mathcal{F}_\epsilon(F_0)$ on another bidder's valuation. Suppose that bidder i bids b_i, her valuation for the item is v_i and $b_j(\cdot)$ is bidder j's bidding strategy for all $j \neq i$. Bidder i's minimum expected payoff from bidding b_i is defined by

$$\min_{F\in\mathcal{F}_\epsilon(F_0)} \int_{v_{-i}\in[0,1]^{n-1}} u_i\Big(b_i, \big(b_j(v_j)\big)_{j\neq i}; v_i\Big) \prod_{j\neq i} dF(v_j).$$

Bidder i's minimum expected payoff maximization problem can be defined as follows:

$$\max_{b_i\in[0,1]} \min_{F\in\mathcal{F}_\epsilon(F_0)} \int_{v_{-i}\in[0,1]^{n-1}} u_i\Big(b_i, \big(b_j(v_j)\big)_{j\neq i}; v_i\Big) \prod_{j\neq i} dF(v_j).$$

By solving this problem for each valuation $v_i \in [0,1]$, bidder i's minimum expected payoff maximizing bid, $b_i(v_i)$, when the other bidders' bidding strategies are $\big(b_j(\cdot)\big)_{j\neq i}$ can be identified. We can say that this bidding function $b_i(\cdot)$ is bidder i's best response against the other bidders' bidding functions $\big(b_j(\cdot)\big)_{j\neq i}$.

In this paper, I adopt the **maxmin Bayesian Nash equilibrium** as the solution concept to investigate the behavior of the bidders in the auction.[3] A strategy profile $\big(b_i^*(\cdot)\big)_{i=1}^n$ is a **maxmin Bayesian Nash equilibrium** if each bidder's bidding strategy is her best response against the other bidders' bidding strategies. That is, for each $i \in N$ and for each $v_i \in [0,1]$,

$$b_i^*(v_i) \in \arg\max_{b_i\in[0,1]} \min_{F\in\mathcal{F}_\epsilon(F_0)} \int_{v_{-i}\in[0,1]^{n-1}} u_i\Big(b_i, \big(b_j^*(v_j)\big)_{j\neq i}; v_i\Big) \prod_{j\neq i} dF(v_j).$$

4 A Maxmin Bayesian Nash Equilibrium

Assume that the seller's reserve price is at least $F_0^{-1}(\epsilon)+\epsilon$. That is,

$$r \geq F_0^{-1}(\epsilon)+\epsilon.$$

Note that $F_0^{-1}(\epsilon)+\epsilon$ is the smallest value of the support of the lower bound distribution in the bidders' prior set.[4] This is assumed because otherwise the bidder with her valuation less than $F_0^{-1}(\epsilon)+\epsilon$ would update her prior set based on the valuation.

Consider bidder i with valuation $v_i \in [0,1]$. Suppose that the profile of the other bidders'

[3]Bose, Ozdenoren, and Pape (2006), Bodoh-Creed (2012), and Wolitzky (2016) use the maxmin Bayesian Nash equilibrium to analyze the decision-making behavior of the agents facing ambiguity.

[4]The lower bound distribution denotes the probability distribution that first-order stochastically dominates all other distributions in the prior set. See Figure 1 for an example.

strategies is $\big(b_j(\cdot)\big)_{j\neq i}$ and each $b_j(\cdot)$ is a strictly increasing function. Then, bidder i's minimum expected payoff maximization problem is defined as follows:

$$\max_{b_i\in[0,1]}\ \min_{F\in\mathcal{F}_\epsilon(F_0)}\ (v_i-b_i)*Pr\,(i \text{ wins the item}).$$

That is,

$$\max_{b_i\in[0,1]}\ \min_{F\in\mathcal{F}_\epsilon(F_0)}\ (v_i-b_i)*Pr\,(b_i\geq b_j(v_j)\ \ \forall j\neq i).^{5}$$

We can make two observations from this maxmin problem. First, bidder i's optimal bid b_i, the minimum expected payoff maximizing bid, is less than or equal to her valuation, v_i, because her expected payoff would be negative otherwise. Second, for each of her bids, $b_i\in[0,1]$, we can find the expected payoff minimizing belief, $F\in\mathcal{F}_\epsilon(F_0)$. From the objective function of the maxmin problem, the expected payoff function, we can obtain that

$$\begin{aligned}&(v_i-b_i)*Pr\,(b_i\geq b_j(v_j)\ \ \forall j\neq i)\\&=(v_i-b_i)*Pr\,(v_j\leq b_j^{-1}(b_i)\ \ \forall j\neq i)\\&=(v_i-b_i)*\prod_{j\neq i}F\big(b_j^{-1}(b_i)\big).\end{aligned}$$

Thus, for given $b_i\in[0,1]$, the expected payoff minimizing belief minimizes bidder i's probability of winning, $\prod_{j\neq i}F\big(b_j^{-1}(b_i)\big)$. Bidder i's probability of winning is a product of cumulative probabilities, $F\big(b_j^{-1}(b_i)\big)$. Thus, a belief minimizing each of these cumulative probabilities is the expected payoff minimizing belief. Therefore, when each bidder's set of priors is given by $\mathcal{F}_\epsilon(F_0)=\{F\in\Delta[0,1]\,|\,F_0(v-\epsilon)-\epsilon\leq F(v)\leq F_0(v+\epsilon)+\epsilon\ \ \forall v\in$

[5] In this formulation, I don't consider the cases of ties for convenience' sake. This does not affect the claim and the results I am going to derive in the paper.

$[0,1]\}$, the expected payoff minimizing belief is F^* satisfying

$$F^*(v) = \begin{cases} 0 & \text{if } v < F_0^{-1}(\epsilon) + \epsilon, \\ F_0(v-\epsilon) - \epsilon & \text{if } F_0^{-1}(\epsilon) + \epsilon \leq v < 1, \\ 1 & \text{if } v \geq 1. \end{cases}$$

We can see that F^* is continuous on $[F_0^{-1}(\epsilon) + \epsilon, 1)$ and discontinuous at 1. Notice also that F^* is bidder i's expected payoff minimizing belief no matter which bid $b_i \in [0,1]$ she chooses.

Example 1. If F_0 is a uniform distribution on $[0,1]$, then the distribution $F^{U*} \in \mathcal{F}_\epsilon(F_0)$ satisfying

$$F^{U*}(v) = \begin{cases} 0 & \text{if } v < 2\epsilon, \\ v - 2\epsilon & \text{if } 2\epsilon \leq v < 1, \\ 1 & \text{if } v \geq 1. \end{cases}$$

is bidder i's expected payoff minimizing belief. In Figure 1, the distribution function on the bottom boundary of the shaded area corresponds to this belief.

Consider bidder i's maxmin expected payoff problem:

$$\max_{b_i \in [0,1]} \min_{F \in \mathcal{F}_\epsilon(F_0)} (v_i - b_i) * \prod_{j \neq i} F\big(b_j^{-1}(b_i)\big).$$

Because F^* obtained above is the expected payoff minimizing belief for any bid b_i, this maxmin expected payoff problem is equivalent to the following problem:

$$\max_{b_i \in [0,1]} (v_i - b_i) * \prod_{j \neq i} F^*\big(b_j^{-1}(b_i)\big).$$

This is bidder i's expected payoff maximization problem when her belief on another bidder's valuation is F^*. From the equivalence of these two problems, it follows that the maxmin Bayesian Nash equilibrium of the first price auction with bidders having sets of priors $\mathcal{F}_\epsilon(F_0)$ is equal to the Bayesian Nash equilibrium of the first price auction with bidders having common prior F^*. There are many previous literature studying Bayesian Nash equilibrium of the first price auction with a common prior and thus, we can find out the maxmin Bayesian Nash equilibrium from the results of those literature. Riley and Samuelson (1981) study the Bayesian Nash equilibrium of the first price auction when bidders have the common belief F that is strictly increasing and differentiable. They show that the equilibrium bidding function is given by

$$b_i(v_i) = v_i - \frac{\int_{v=r}^{v_i} (F(v))^{n-1} dv}{(F(v_i))^{n-1}} \tag{1}$$

for $v_i \geq r$ where r is the seller's reserve price. Monteiro (2009) generalizes this result and identifies the Bayesian Nash equilibrium when the common prior F has discontinuities. He shows that the equilibrium bidding strategy is equal to (1) at the continuous points of F and a mixed strategy at the discontinuities of F. The expected payoff minimizing belief, F^*, in our paper has one discontinuity at $v = 1$. Thus, we can find out a maxmin Bayesian Nash equilibrium of the first price auction as follows by using the result of Monteiro (2009).

Proposition 1. Consider a first price sealed-bid auction with the seller's reserve price r satisfying $r \geq F_0^{-1}(\epsilon) + \epsilon$. Suppose that each bidder has a set of priors $\mathcal{F}_\epsilon(F_0)$ about another bidder's valuation for the item. Then, a profile of mixed strategies $\big(\mu_i(\cdot)\big)_{i=1}^n$ is a **maxmin Bayesian Nash equilibrium** of the auction if for each $i \in \{1, \ldots, n\}$,

$$\mu_i(v_i) = \begin{cases} \text{pure strategy,} \quad b_i^*(v_i) = v_i - \dfrac{\int_{v=r}^{v_i} \big(F_0(v-\epsilon)-\epsilon\big)^{n-1} dv}{\big(F_0(v_i-\epsilon)-\epsilon\big)^{n-1}} & \text{if } v_i \in [r, 1), \\ \text{mixed strategy,} \quad G & \text{if } v_i = 1, \end{cases}$$

where $G : \big[b_i^{F^*}(1-), b_i^{F^*}(1)\big] \to \big[0, 1\big]$ is a cumulative distribution function satisfying

$$G(b) = \frac{F_0(1-\epsilon)-\epsilon}{1-(F_0(1-\epsilon)-\epsilon)}\Bigg(-1 + \bigg(\frac{1-b_i^{F^*}(1-)}{1-b}\bigg)^{\frac{1}{n-1}}\Bigg),$$

F^* is the expected payoff minimizing prior, and

$$b_i^{F^*}(v_i) = v_i - \frac{\int_{v=r}^{v_i} \big(F^*(v)\big)^{n-1} dv}{\big(F^*(v_i)\big)^{n-1}} \qquad \text{for } v_i \in [r, 1].$$

Remark 1. In the maxmin Bayesian Nash equilibrium, a bidder plays the mixed strategy G only when her valuation is equal to 1. Because the true distribution F_0 is continuous, the event that the bidder's valuations for the item is equal to 1 has measure 0. Thus, we focus on the bidder's pure strategy, $v_i - \dfrac{\int_{v=r}^{v_i} \big(F_0(v-\epsilon)-\epsilon\big)^{n-1} dv}{\big(F_0(v_i-\epsilon)-\epsilon\big)^{n-1}}$, from this point. Let $b_i^*(v_i)$ denote this pure strategy for $v_i \in [r, 1)$.

Example 1. (continued.) Suppose that the true distribution F_0 is a uniform distribution on $[0, 1]$, that is, $F_0(v) = v$ for $v \in [0, 1]$. Then, bidder i's maxmin Bayesian Nash equilibrium bidding strategy for $v_i \in [r, 1)$ is given by

$$b_i^*(v_i) \;=\; v_i - \frac{\int_{v=r}^{v_i} (v - 2\epsilon)^{n-1}\, dv}{(v_i - 2\epsilon)^{n-1}} \;=\; \frac{n-1}{n} v_i + \frac{2\epsilon}{n} + \frac{1}{n} \frac{(r - 2\epsilon)^n}{(v_i - 2\epsilon)^{n-1}}.$$

If the bidders don't have any ambiguity about the other bidders' valuations ($\epsilon = 0$), then each bidder's Bayesian Nash equilibrium bidding strategy is $b_i(v_i) = \dfrac{n-1}{n} v_i + \dfrac{r^n}{n v_i^{n-1}}$.

5 Changes in Bidders' Ambiguity Level

From the maxmin Bayesian Nash equilibrium of the first price auction we obtained in Proposition 1, we can study how each bidder's equilibrium bidding behavior changes as her level of ambiguity changes. The higher value of ϵ implies the larger set of beliefs, $\mathcal{F}_\epsilon(F_0)$, and thus, the higher level of ambiguity that each bidder has. It can be shown that each bidder bids higher in response to the higher level of ambiguity if the probability distribution of the bidders' valuations, F_0, satisfies a certain condition.

Definition 1. The distribution F satisfies the **monotone inverse hazard rate condition** if $\dfrac{f(v)}{F(v)}$ is non-increasing in v.

Then, we can obtain the following result:

Proposition 2. Suppose that F_0 satisfies the monotone inverse hazard rate condition. Then, each bidder with her valuation for the item $v_i \in (r, 1)$ submits a strictly higher bid in response to an increased level of ambiguity.

Proof. From the Proposition 1, it follows that bidder i's maxmin Bayesian Nash equilibrium bidding strategy is

$$b_i^*(v_i) = v_i - \frac{\int_{v=r}^{v_i} \big(F_0(v-\epsilon)-\epsilon\big)^{n-1}\,dv}{\big(F_0(v_i-\epsilon)-\epsilon\big)^{n-1}}.$$

If we take the derivative of the function with respect to ϵ, then we obtain that

$$\frac{db_i^*(v_i)}{d\epsilon} = \frac{\int_{v=r}^{v_i} (n-1)\big(F_0(v-\epsilon)-\epsilon\big)^{n-2}\big(F_0(v_i-\epsilon)-\epsilon\big)^{n-1}\big(f_0(v-\epsilon)+1\big)\,dv}{\big(F_0(v_i-\epsilon)-\epsilon\big)^{2n-2}}$$
$$- \frac{\int_{v=r}^{v_i} (n-1)\big(F_0(v-\epsilon)-\epsilon\big)^{n-1}\big(F_0(v_i-\epsilon)-\epsilon\big)^{n-2}\big(f_0(v_i-\epsilon)+1\big)\,dv}{\big(F_0(v_i-\epsilon)-\epsilon\big)^{2n-2}}$$

$$= \frac{\begin{array}{c}\int_{v=r}^{v_i}(n-1)\big(F_0(v-\epsilon)-\epsilon\big)^{n-2}\big(F_0(v_i-\epsilon)-\epsilon\big)^{n-2} \\ *\Big[\big(F_0(v_i-\epsilon)-\epsilon\big)\big(f_0(v-\epsilon)+1\big)-\big(F_0(v-\epsilon)-\epsilon\big)\big(f_0(v_i-\epsilon)+1\big)\Big]dv\end{array}}{\big(F_0(v_i-\epsilon)-\epsilon\big)^{2n-2}}. \quad (2)$$

Consider the following term in the brackets in (2):

$$\Big[\big(F_0(v_i-\epsilon)-\epsilon\big)\big(f_0(v-\epsilon)+1\big)-\big(F_0(v-\epsilon)-\epsilon\big)\big(f_0(v_i-\epsilon)+1\big)\Big].$$

It is given that $v \leq v_i$. It can be shown that the value of this term is strictly positive if $v < v_i$. There are two possible cases to consider: $f_0(v-\epsilon) \geq f_0(v_i-\epsilon)$ and $f_0(v-\epsilon) < f_0(v_i-\epsilon)$. Consider the case that $f_0(v-\epsilon) \geq f_0(v_i-\epsilon)$. Because F_0 is strictly increasing, it follows that $F_0(v_i-\epsilon)-\epsilon > F_0(v-\epsilon)-\epsilon$. Thus, we can obtain that $\big(F_0(v_i-\epsilon)-\epsilon\big)\big(f_0(v-\epsilon)+1\big)-\big(F_0(v-\epsilon)-\epsilon\big)\big(f_0(v_i-\epsilon)+1\big) > 0$. Consider the other case that $f_0(v-\epsilon) < f_0(v_i-\epsilon)$. Because F_0 satisfies the monotone inverse hazard rate condition, it follows that $\frac{f_0(v-\epsilon)}{F_0(v-\epsilon)} \geq \frac{f_0(v_i-\epsilon)}{F_0(v_i-\epsilon)}$, that is, $F_0(v_i-\epsilon)f_0(v-\epsilon)-F_0(v-\epsilon)f_0(v_i-\epsilon) \geq 0$.

Thus, we obtain that

$$
\begin{aligned}
&F_0(v_i-\epsilon)f_0(v-\epsilon)-F_0(v-\epsilon)f_0(v_i-\epsilon)\geq 0\\
&\Rightarrow\ \big(F_0(v_i-\epsilon)-\epsilon\big)f_0(v-\epsilon)-\big(F_0(v-\epsilon)-\epsilon\big)f_0(v_i-\epsilon)>0\\
&\Rightarrow\ \big(F_0(v_i-\epsilon)-\epsilon\big)\big(f_0(v-\epsilon)+1\big)-\big(F_0(v-\epsilon)-\epsilon\big)\big(f_0(v_i-\epsilon)+1\big)>0.
\end{aligned}
$$

The second inequality is obtained from the supposition that $f_0(v-\epsilon)<f_0(v_i-\epsilon)$. The third inequality is obtained from strictly increasing F_0. By investigating the two possible cases, we have that the value of the term, $\big(F_0(v_i-\epsilon)-\epsilon\big)\big(f_0(v-\epsilon)+1\big)-\big(F_0(v-\epsilon)-\epsilon\big)\big(f_0(v_i-\epsilon)+1\big)$, is strictly positive if $v<v_i$. Therefore, the value of the integral of (2) is strictly positive.■

Example 1. (continued.) If F_0 is a uniform distribution on $[0,1]$, then its inverse hazard rate function, $\frac{1}{v}$, is decreasing in v. Therefore, we can apply Proposition 2 and say that each bidder responds to increased level of ambiguity with a higher bid when F_0 is a uniform distribution.

Because the bidders increase their bids in response to an increased level of ambiguity, the seller's expected revenue from the auction also increases.

Proposition 3. Suppose that F_0 satisfies the monotone inverse hazard rate condition. Then, the seller's expected revenue from the auction when bidders face ambiguity is greater than the one when bidders don't face ambiguity. Moreover, the seller's expected revenue increases as the level of ambiguity faced by the bidders increases.

Proof. Suppose that each bidder in the auction has a prior set $\mathcal{F}_\epsilon(F_0)$ about the other bidders' valuations. Let $b^{\epsilon*}(\cdot)$ denote the maxmin Bayesian Nash equilibrium bidding strategy of each bidder we obtained in Proposition 1. Then, the seller's expected revenue is defined by

$$R(\epsilon) = \int_{v=r}^{1} b^{\epsilon *}(v)\, n\big(F_0(v)\big)^{n-1} f_0(v)\, dv.$$

Consider two numbers, $\epsilon_1 \in [0,1]$ and $\epsilon_2 \in [0,1]$, satisfying $\epsilon_1 < \epsilon_2$. Note that a bidder having prior set $\mathcal{F}_{\epsilon_2}(F_0)$ faces the higher level of ambiguity than a bidder having prior set $\mathcal{F}_{\epsilon_1}(F_0)$. Note also that if $\epsilon_1 = 0$, then the prior set $\mathcal{F}_{\epsilon_1}(F_0)$ is a singleton and it implies that the bidder does not have ambiguity. Because $\epsilon_1 < \epsilon_2$, it follows that $b^{\epsilon_1 *}(v) < b^{\epsilon_2 *}(v)$ for all $v \in (r, 1)$ from the result of Proposition 2. Therefore, we can obtain that $R(\epsilon_1) < R(\epsilon_2)$.■

6 A comparison with the second price auction

The first price auction is compared to another popular auction format, the second price auction, when bidders have ambiguity about the probability distribution from which the other bidders' valuations for the item are drawn.

It can be shown that the bidders in the second price auction have an incentive to bid their own valuations even when they face ambiguity. Consider the second price auction with bidders facing no ambiguity. In this case, there is a well-known result that truthful bidding from each bidder forms a dominant strategy equilibrium. Since it is a dominant strategy, each bidder's incentive for truthful bidding does not depend on her belief about the other bidders' valuations. Thus, even when the bidders face ambiguity and have multiple beliefs, truthful bidding forms a dominant strategy equilibrium in the second price auction.

The bidders in the second price auction always bid their true valuations, no matter whether they have ambiguity or not about the distribution of the others' valuations. Thus, the seller's expected revenue from the second price auction also does not depend on whether the bidders face ambiguity or not. However, we know from proposition 3 that the seller's expected revenue from the first price auction increases as the bidders' ambiguity level in the auction increases. When bidders don't face ambiguity, there is a well-known revenue equivalence result between first price and second price auctions. From these results on the seller's expected revenue from two auction formats, we can obtain the following result:

Proposition 4. Consider a first price auction and a second price auction with the seller's reserve price r. Suppose that the true distribution, F_0, from which each bidder's valuation is drawn satisfies the monotone inverse hazard rate condition, the bidders face ambiguity about the distribution and they are ambiguity averse, and each bidder's set of priors is $\mathcal{F}_\epsilon(F_0)$. Then the seller can obtain the higher expected revenue from the first price auction than the second price auction. Moreover, the difference in expected revenues from two auction formats increases as the level of ambiguity faced by the bidders increases.

7 Conclusion

This paper analyzes the first price auction where each bidder faces ambiguity about the probability distribution from which the other bidders' valuations are independently drawn and is ambiguity averse. The maxmin expected utility model with multiple priors axiomatized by Gilboa and Schmeidler (1989) is used to solve the bidders' optimal bidding problems. The bidders' equilibrium bidding functions and the seller's expected revenue from the auction are identified. Moreover, it is shown that the bidders' equilibrium bids and the seller's expected revenue increase as the bidders' ambiguity level increases. It is also determined that the seller's expected revenue from the first price auction is greater than that of the second price auction when the bidders face ambiguity.

8 Future research directions

Asymmetry in bidders' ambiguity levels. I assumed that the level of ambiguity is the same for all bidders. As a next step, I plan to relax this assumption and assign different ambiguity levels to bidders. Under these asymmetric ambiguity levels, studying the differences between the bidders' bidding strategies based on their ambiguity levels is a future research direction.

Minimax regret decision rule. I used the maxmin expected utility decision rule. There is another decision rule, the minimax regret model axiomatized by Stoye (2011), that explains the decision-making behavior of agents with ambiguity. Exploring the bidders' equilibrium bidding strategies and the seller's revenue using this decision rule is another future research topic.

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