Implementation by vote-buying mechanisms*

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Abstract

A vote-buying mechanism is such that each agent chooses a quantity of votes $x$ to cast for an alternative at a cost $c(x)$, and the outcome is determined by the total number of votes cast for each alternative. In the context of binary decisions, we prove that the welfare optima that can be implemented by vote-buying mechanisms in large societies are parameterized by a positive parameter $\rho$, which measures the importance of preference intensities on the social choice: The limit with $\rho = 0$ is in line with the majoritarian principle, $\rho = 1$ corresponds to utilitarianism, and $\rho \to \infty$ agrees with the Rawlsian optimum. We show that any vote-buying mechanism with limit cost elasticity

$$\lim_{x \to 0} \frac{c'(x)x}{c(x)} = 1 + 1/\rho$$

implements the welfare optimum defined by $\rho$. The utilitarian efficiency of quadratic voting (Lalley and Weyl, 2016) follows as a special case.

Keywords: implementation; mechanism design; welfare optimum; social choice correspondence; efficiency; utilitarianism.

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1 Introduction

Consider a binary collective choice problem: a society must choose one of two alternatives. We study the mechanisms that the society can use to make its choice, and their welfare properties.

The welfare properties of a given mechanism depend on the welfare notion we use to evaluate the mechanism. Different normative theories disagree on whether society should take into account the intensity of individual preferences to make its choice. According to Dahl’s (1989), a democracy must satisfy “voting equality”: each citizen’s vote must be weighed equally, with no role for preference intensity. According to the majoritarian criterion advocated by political theorists such as Locke (1689) and Spitz (1984), the welfare optimum is the alternative preferred by a majority of voters. Majority rule, which follows the majoritarian principle, aggregates purely ordinal preferences, disregarding preference intensity, and it is the unique rule that satisfies May’s (1952) axioms for preference aggregation. In contrast, the maximin principle (Rawls 1971) considers only intensity of preferences: it declares that the welfare optimum is the alternative that maximizes the utility of the least satisfied individual. Majoritarianism and Rawlsian maximin are the two extremes of a class composed of a continuum of intermediate normative criteria, including utilitarianism (Stuart Mill 1863). Each normative criteria in this class takes preference intensity into account to a different degree.

A mechanism is optimal for a society that embraces a given normative criterion if the alternative chosen by the mechanism is always the welfare optimum according to the society’s criterion. Note that to the extent that different societies embrace different normative criteria, each society needs a mechanism tailored to its own normative criterion. We address this need: for each normative criterion in our infinite class, we propose a mechanism that asymptotically achieves the goal of always choosing this criterion’s welfare optimum as the society becomes large.

The mechanisms we propose are “vote-buying” mechanisms: each subject can express her intensity of preference by acquiring any quantity of votes $x$ for either alternative, at a pre-announced cost, $c(x)$, and the social choice is determined by the total number of votes cast
for each alternative.

Lalley and Weyl (2016) introduced a mechanism that asymptotically attains our goal for one specific normative criterion: vote-buying with a quadratic cost of votes implements the utilitarian optimum—the option that maximizes the unweighted sum of the individual utilities. We extend their result, proposing a solution for any normative criterion defined by a positive parameter \( \rho \in \mathbb{R}_{++} \) which measures the importance of preference intensity.

The welfare optimum for a given \( \rho \) is alternative \( A \) if \( \sum_{i=1}^{n} \text{sgn}(v_i)|v_i|^\rho > 0 \), and alternative \( B \) if \( \sum_{i=1}^{n} \text{sgn}(v_i)|v_i|^\rho < 0 \); where \( v_i \) is the amount of real wealth that subject \( i \) would trade to change the social choice from a random coin toss to \( A \) with certainty, and \( \text{sgn}(v_i) \) is the sign (positive or negative) of \( v_i \). That is, \( v_i \) measures how much the individual cares that the social choice be \( A \) and not \( B \) (naturally, subjects who prefer \( B \) have a negative valuation). The set, indexed by \( \rho \in \mathbb{R}_{++} \), of all such normative criteria is defined by a collection of appealing axioms (Bergson 1936, Roberts 1986, Moulin 1988, Eguia and Xefteris 2017).\(^1\)

The majoritarian principle is at lower limit of this set, \( \rho = 0 \). Utilitarianism is the criterion defined by parameter \( \rho = 1 \): the utilitarian welfare optimum is alternative \( A \) if \( \sum_{i=1}^{n} v_i > 0 \). At the higher limit of the set, the welfare optimum according is \( \rho = \infty \) is the alternative preferred by the agent whose valuation has the highest absolute value. Throughout the class of normative criteria, the welfare optimum according to a smaller \( \rho \) is highly influenced by the number of subjects who support each alternative, and less so by their intensity, while when \( \rho \) is large the welfare optimum better reflects the preferences of the individuals whose well-being is greatly affected by the decision.

We say that a normative criterion is asymptotically implemented by a given mechanism if the probability that the choice using this mechanism is the welfare optimum of the given criterion converges to one as society becomes large. We characterize the class of normative criteria that are asymptotically implementable by vote-buying mechanisms: we show that any vote-buying mechanism with limit cost elasticity \( \lim_{x \to 0} \frac{c'(x)x}{c(x)} = 1 + 1/\rho \) asymptotically

\(^1\)The axioms are: anonymity, neutrality, monotonicity, continuity, independence of unconcerned agents, and scale-invariance.
implements the normative criterion defined by $\rho$; and normative criteria outside this class are not asymptotically implementable.

We stress that, as in Lalley and Weyl (2016), the vote-buying mechanisms that we consider are robust in the sense they do not require the mechanism designer to know at the time she designs the mechanism, the particular features of the society, such as the number of individuals, the exact distribution of types from which individual preferences are drawn, or the importance of the choice under consideration. Hence, we interpret the proposed vote-buying mechanisms as institutions which implement the $\rho$ welfare optima in large societies, regardless of changes in distributional parameters.

**Literature Review**

Our work builds on Lalley and Weyl (2016), and the literature on quadratic voting that has developed around it, including Goeere and Zhang (2017), and the special issues 1 and 2 of Volume 172 of the journal *Public Choice*, edited by Weyl and Posner (2017), in their entirety.² Like this literature, we propose vote-buying mechanisms to implement social choice correspondences in binary collective choice problems. Unlike it, we look beyond utilitarianism: we let the mechanism designer choose within a large menu of welfare notions, and we offer a mechanism that asymptotically implements the designer’s welfare optimum. The menu of welfare notions we consider follows the axiomatic foundation laid by the research on cardinal welfare summarized by Moulin (1988).

Our work, like all the literature on vote-buying mechanisms, has deeper roots in classic mechanism design. The VCG mechanism (Vickrey 1961, Clark 1971 and Groves 1973) satisfies utilitarian efficiency, but is not budget-balanced. We want a budget-balanced mechanism. The mechanisms by Arrow (1979) and AGV (D’Aspremont and Gerard-Varet 1979) are budget-balanced and attain utilitarian efficiency by requiring each agent to pay the expected externality of her choices, but to calculate this expected externality, the designer must know population parameters such as the distribution from which individual preferences are drawn.

²Of particular interest to us are the entries on robustness to collusion (Weyl 2017), agenda-setting (Patty and Penn 2017) and turnout (Kaplov and Kominers 2017).
The designer we have in mind does not have this information. Put differently: the AGV mechanism works when it is designed specifically for a particular society with known population parameters at a specific point in time; whereas, we propose a mechanism that works for many societies that share a common normative principle but differ in their population parameters, so that the mechanism is robust over time as the values of exogenous parameters change.

Related approaches to gauge intensity of preferences through voting involve allowing decentralized markets for votes (Dekel, Jackson and Wolinski 2008, Casella, Llorente-Saguer and Palfrey 2012). A competitive equilibrium in a market for votes is very similar to our special case with parameter \( \rho = \infty \): the cost of votes is linear, and the agent who cares most about the decision buys most votes.

Our results generalize Lalley and Weyl’s (2016) intuition that quadratic voting asymptotically attains utilitarian efficiency, but the two models are not nested: to obtain simpler and shorter proofs, we make assumptions on the payoff function that are substantially similar, but technically distinct. A greater conceptual difference between Lalley and Weyl (2016)’s approach and ours is that they study the properties of a particular mechanism; whereas, our theory is an exercise in Bayesian implementation (Jackson 1991): given a desired social choice correspondence, we seek a mechanism such that in any equilibrium, the outcome coincides with the desired social choice, for any realization of preferences. From the literature on implementation via bounded mechanisms (Jackson 1992 and Jackson, Palfrey and Srivastava 1994) we echo the principle that mechanisms should not be too complex. From virtual implementation (Matsushima 1988, Abreu and Sen 1991), we inherit the idea that if exact implementation is not possible, a second best is to settle for choosing the desired outcome with probability arbitrarily close to one.

Lalley and Weyl (2016) provide a more extensive discussion of quadratic voting, its precedents and related literature, its heuristic intuition, and potential challenges to its roll-out in real world applications. We refer the interested reader to their insightful discussion, and do not replicate it here.
2 The Formal Framework

Summary. A set of agents must make a binary social choice. The decision is made via a vote-buying mechanism: agents purchase votes, and the alternative with the most votes is chosen. We characterize the set of social choice correspondences that are implementable by these vote buying mechanisms.

Players. For any \( n \in \mathbb{N}\backslash\{1\} \), let \( N^n \) be a society composed of a set of agents \( \{1, \ldots, n\} \). For each \( n \in \mathbb{N}\backslash\{1\} \), for any \( i \in N^n \) and for any agent-specific variable \( y_i \), define \( y_{-i} \equiv (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \), and define \( y_{N^n} \equiv (y_1, \ldots, y_n) \).

Social choice set. This society must make a binary choice over \( \{A, B\} \). Let the social decision \( d \in \{A, B\} \) denote the alternative chosen.

Wealth. Each agent \( i \in N^n \) is endowed with initial wealth \( \bar{w}_i \in \mathbb{R}_+ \). Let \( w_i \in \mathbb{R} \) denote the final wealth of agent \( i \) in nominal terms, after the social decision is made. Let \( p \in \mathbb{R}_{++} \) be a price parameter, an exogenously given characteristic of the society. Let \( \frac{w_i}{p} \) and \( \frac{w_i}{p} \) denote the real initial wealth and real final wealth of agent \( i \).

Assumption 1. Agents face no budget constraints (final wealth can be negative).

Equivalently, we could assume that the initial wealth of each agent is sufficiently large relative to the importance of the social choice that no undominated expense related to the social choice would hit the budget constraint.\(^3\)

Outcomes. The set of outcomes is \( \{A, B\} \times \mathbb{R}^n \), where the first component is the social choice \( x \), and the second is the vector \( \frac{w_{N^n}}{p} \) of real final wealth. Let \( \mathcal{M}(\{A, B\} \times \mathbb{R}^n) \) denote the set of probability measures over the set of outcomes, allowing us to consider stochastic outcomes.

\(^3\)This assumption is not restrictive, because –as we will show- if the society is large, the amount voters spend to purchase votes converges to zero, so for any given positive budget constraint, in a sufficiently large society, the constraint would not bind.
Let $\mu$ denote an arbitrary probability measure in $\mathcal{M}(\{A, B\} \times \mathbb{R}^n)$ and let $L$ denote a simple lottery over $\{A, B\} \times \mathbb{R}^n$ (a probability measure that assigns strictly positive probability only to finitely many outcomes).

Preferences. Each agent $i \in N^n$ has a (complete, transitive) preference order $\succsim_i$ over $\mathcal{M}(\{A, B\} \times \mathbb{R}^n)$. Let $\sim_i$ be the associated indifference relation and note that $\succsim_{N^n} \equiv (\succsim_1, \ldots, \succsim_n)$ denotes the preference profile.

We assume that for each $i \in N^n$, the preference order $\succsim_i$ is continuous and it satisfies independence over decomposition of lotteries, so that it can be represented by a continuous utility function in expected utility form, following von Neumann and Morgenstern’s (1944) Expected Utility theorem. Further, we assume that each $i \in N^n$ cares only about the social decision $d \in \{A, B\}$, and about her final real wealth $\frac{w_i}{p}$, (and not about the wealth of other agents), that $\succsim_i$ is separable (Fishburn [13]) and strictly monotonic on real final wealth, and that agent $i$ is risk neutral with respect to final real wealth. For convenience, we provide formal statements of these standard assumptions in the Appendix (Assumptions 2-6).

Together, these assumptions on preferences imply that $\succsim_i$ is representable by an additively separable, quasilinear expected utility function, in which the first term of the summation is the expected utility from the social decision, and the second term is the expected final wealth.

Valuation of the Social Choice. Let $[-1, 1]$ denote the set of possible attitudes toward choice $A$, from least favorable (-1) to most favorable (+1). Let $\mathcal{F}$ be the set of all cumulative distributions over $[-1, 1]$ that are symmetric around 0, twice continuously differentiable, have strictly positive density over the domain, and no mass at any point. Let $F \in \mathcal{F}$ denote one such arbitrary distribution, and let $f$ denote its density.

Let $\tilde{\theta}$ be a random variable that follows distribution $F$. Assume that for any $n \in \mathbb{N}\{1\}$, for each $i \in N^n$, attitude $\theta_i$ is an independent draw of $\tilde{\theta}$. Assume that for each $n \in \mathbb{N}\{1\}$ and for each $i \in N^n$, $\theta_i$ is privately observed.
Let $\gamma \in \mathbb{R}_{++}$ be a parameter that represents the importance to society of the social choice under consideration. This is a society-wide parameter, observed by all agents.

**Assumption 7.** For any $n \in \mathbb{N}\backslash\{1\}$, for any $i \in N^n$ and for any $w_i \in \mathbb{R}$, \(\left( A, \frac{w_i}{p} - \gamma \theta_i \right) \sim_i \left( B, \frac{w_i}{p} + \gamma \theta_i \right)\).

That is, each agent is indifferent between an outcome with the most preferred social choice and a real wealth loss of $\gamma \theta_i$ and an outcome with the least preferred social choice and a real wealth gain of $\gamma \theta_i$. Put differently, the social choice is worth $2\gamma \theta_i$ units of real wealth to agent $i$.

**Actions.** For any $n \in \mathbb{N}\backslash\{1\}$, each agent $i \in N^n$ chooses an action $a_i \in \mathbb{R}$. Strictly positive actions are interpreted as in favor of $A$, and strictly negative ones, as against $A$ (or, equivalently, in favor of $B$).

**Vote buying mechanisms.** A vote buying mechanism is defined by a cost function $c : \mathbb{R} \rightarrow \mathbb{R}_+$, such that for any $n \in \mathbb{N}\backslash\{1\}$, and for any $x \in \mathbb{R}$, any agent $i \in N^n$ who chooses action $a_i = x$ pays a cost $c(x)$. All payments are redistributed equally among all other agents, so given a vector of actions $a_{N^n} \in \mathbb{R}^n$, each agent $i \in N^n$ obtains a net transfer $-c(a_i) + \sum_{j \in N^n \backslash \{i\}} \frac{c(a_j)}{n-1}$.

A perfect execution of a mechanism $c$ would entail society choosing $d = A$ if $\sum_{j \in N^n} a_j > 0$ and $d = B$ if $\sum_{j \in N^n} a_j < 0$. However, we assume that the mechanism designer anticipates that executing any mechanism entails some small element, so that the mapping from actions to outcomes is stochastic and the probability that $d = A$ is increasing in $\sum_{j \in N^n} a_j$, but is not a step function.

Formally, for any $a_{N^n}$, we assume that the probability that $d = A$ is $G\left( \sum_{j \in N^n} a_j \right)$, where $G : \mathbb{R} \rightarrow [0,1]$ is a strictly increasing, twice continuously differentiable, cumulative distribution function, and $g : \mathbb{R} \rightarrow [0,1]$ is its associated density function. We assume that $G \in \mathcal{G}$, where $\mathcal{G}$ is the class of all strictly increasing, twice continuously differentiable functions from $\mathbb{R} \rightarrow [0,1]$ such that for any $\tilde{G} \in \mathcal{G}$ with density $\tilde{g}$ and derivative of the density $\tilde{g}'$:
i) \( \tilde{G}(x) - \frac{1}{2} = \frac{1}{2} - \tilde{G}(-x) \) for any \( x \in \mathbb{R}_{++} \);

ii) \( \lim_{x \to -\infty} \tilde{G}(x) = 0 \) and \( \lim_{x \to -\infty} \tilde{g}(x) = 0 \);

iii) \( \exists \varepsilon \in \mathbb{R}_{++} \) such that \( \lim_{x \to -\infty} \frac{\tilde{g}(x+\varepsilon)}{\tilde{g}(x)} \in \mathbb{R} \) and \( \lim_{x \to -\infty} \frac{\tilde{g}(x+\varepsilon)}{\tilde{g}(x)} \in \mathbb{R} \forall \varepsilon \in (0, \varepsilon) \).

Condition (i) is neutrality. Condition (ii) is a responsiveness condition: if the vote margin is sufficiently large, the outcome is the one with the vote advantage with probability arbitrarily close to one. Condition iii) requires the tails of the density not to drop to zero too steeply. The set \( \mathcal{G} \) contains, among others, all Student-t distributions.

**Strategies.** Each agent \( i \) in society \( N^n \), with price index \( p \in \mathbb{R}_{++} \), with wealth distribution \( w_{N^n} \in \mathbb{R}_{++}^n \), facing a social decision of importance \( \gamma \in \mathbb{R}_{++} \) to be decided according to mechanism \( c \in C \) under uncertainty \( G \in \mathcal{G} \), and taking into account that the ex-ante distribution of attitudes toward the decision is given by distribution \( F \), chooses to purchase a quantity of votes \( a_i \in \mathbb{R} \) as a function of her realization \( \theta_i \in [-1, 1] \) of her own attitude toward the decision. Therefore, taking \( n, F, \gamma, w_{N^n}, p, c \) and \( G \) as given, a pure strategy is a mapping \( s : [-1, 1] \to \mathbb{R} \). Let \( S \) be the set of all feasible pure strategies. Let \( s(\theta) \) be the action taken given \( \theta \) according to strategy \( s \), always given \( n, F, \gamma, w_{N^n}, p, c \) and \( G \). Let \( \sigma : [-1, 1] \to \mathcal{M}(\mathbb{R}) \) denote a mixed strategy. Let \( \Sigma \) denote the set of all mixed strategies. Let \( s_i = s \) denote that agent \( i \) chooses strategy \( s \).

**Definition 1** We say that a strategy \( s \) is neutral if \( s(-\theta) = -s(\theta) \) for any \( \theta \in [-1, 1] \). We say \( s \) is monotone if \( \frac{d s}{d \theta} \geq 0 \).

**Utilities.** Given a society \( N^n \) with \( (n, F, w_{N^n}, \gamma, p, G) \in \mathbb{N} \setminus \{1\} \times \mathcal{F} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathcal{G} \) and given mechanism \( c \in C \), for any agent \( i \in N^n \) with preference order \( \succ_i \) over \( \mathcal{M}(\{A, B\} \times \mathbb{R}^n) \), we can compute the expected utility of agent \( i \) as a function of her type (initial wealth \( \bar{w}_i \) and attitude \( \theta_i \)), her strategy \( s_i \) and the strategy profile of every other player \( s_{-i} \). Let \( EU_i : \mathbb{R}_+ \times [-1, 1] \times S^n \to \mathbb{R} \) denote the expected utility of agent \( i \). Then, given \( (n, F, \gamma, w_{N^n}, p, c, G) \),
for any $\theta_i \in [-1, 1]$ and $s_{N^n} \in S^n$, $EU_i[\theta_i, s_{N^n}]$ is equal to

$$
\gamma \theta_i \left( \frac{2}{\theta_{-i} \in [-1, 1]} \int_{n=1}^{n-1} \left( \prod_{j \in N^n \setminus \{i\}} f(\theta_j) \right) G \left( s_i(\theta_i) + \sum_{j \in N^n \setminus \{i\}} s_j(\theta_j) \right) d\theta_{-i} - 1 \right) + \bar{w}_i - c(s_i(\theta_i)) + \frac{1}{n-1} \sum_{j \in N^n \setminus \{i\}} \int_{\theta_j \in [-1, 1]} f(\theta_j) c(s_j(\theta_j)) d\theta_j.
$$

(1)

**Game.** For each tuple $(n, F, \gamma, \bar{w}_{N^n}, p, c, G) \in \mathbb{N} \setminus \{1\} \times F \times \mathbb{R}_{++} \times \mathbb{R}_+ \times \mathbb{R}_{++} \times C \times G$, let $\Gamma^{(n, F, \gamma, \bar{w}_{N^n}, p, c, G)}$ denote the game played by the $n$ players in society $N^n$, with strategy set $S$ for each agent, and expected utility given by $EU_i$ in expression 1 for each $n \in \mathbb{N} \setminus \{1\}$ and each $i \in N^n$.

**Equilibrium.** For any tuple $(n, F, \gamma, \bar{w}_{N^n}, p, c, G) \in \mathbb{N} \setminus \{1\} \times F \times \mathbb{R}_{++} \times \mathbb{R}_+ \times \mathbb{R}_{++} \times C \times G$, let $BNE^{(n, F, \gamma, \bar{w}_{N^n}, p, c, G)} \subseteq \Sigma^n$ denote the set of Bayes Nash Equilibria of game $\Gamma^{(n, F, \gamma, \bar{w}_{N^n}, p, c, G)}$. We are interested in the subset of symmetric pure $BNE$, in which each player plays the same pure, neutral, monotone strategy. Let $E^{(n, F, \gamma, p, c, G)} \subseteq S$ be the set of symmetric, pure, neutral and monotone equilibrium strategies.\(^4\)

**Sequence of Societies.** We consider a sequence of societies $\{N^n\}_{n=2}^{\infty}$, where, for each $n \in \mathbb{N} \setminus \{1\}$, $N^{n+1} = N^n \cup \{n + 1\}$. We will establish results for sufficiently large societies. Note that aside from the size $n \in \mathbb{N}$, $(F, \gamma, \bar{w}_{N^n}, p, G)$ are the characteristics that identify a particular society.

**Social welfare functions.** A social welfare function $W: \bigcup_{n=2}^{\infty} \mathbb{R}^n \rightarrow \mathbb{R}$ is a mapping from a vector of valuations $\gamma \theta_{N^n}$ of any length $n \in \mathbb{N} \setminus \{1\}$, to the real line, where we interpret $W(\gamma \theta_{N^n})$ as the social welfare derived from choosing alternative $A$ given the profile of valua-

\(^4\)We drop the superindex $w_{N^n}$ because the equilibria, as we show below, will not depend on $w_{N^n}$.
tions $\gamma \theta_{N^n}$, and $-W(\gamma \theta_{N^n})$ as the social welfare of choosing alternative $B$.

Let $\text{sgn} : \mathbb{R} \rightarrow \{-1, 0, 1\}$ be the sign function, defined by $\text{sgn}(x) = -1$ if $x < 0$, $\text{sgn}(x) = 0$ and $\text{sgn}(x) = 1$ if $x > 0$. For each $\rho \in \mathbb{R}^{++}$, define the Bergson social welfare function $W_\rho$ by

$$W_\rho(\gamma \theta_{N^n}) = \sum_{i \in N^n} \text{sgn}(\theta_i)|\gamma \theta_i|^\rho.$$ 

and define the set of Bergson welfare functions $W \equiv \bigcup_{\rho \in \mathbb{R}^{++}} \{W_\rho\}$. This set of Bergson welfare functions (Burk 1936), parameterized by $\rho$, is characterized by the following set of axioms: anonymity, neutrality, monotonicity, continuity, independence of unconcerned agents and scale-invariance (Eguia and Xefteris 2017, based on Moulin 1988).\(^5\) Note that $\rho = 1$ is utilitarian welfare.

**Efficiency.** Given a welfare function $W$, a social decision is efficient if it is the welfare maximizer according to $W$; that is, it is $A$ if $W(\gamma \theta_{N^n}) > 0$ and $B$ if $W(\gamma \theta_{N^n}) < 0$.

Given any welfare function $W$, we say that a mechanism $c$ is efficient according to $W$ if for any society, for any equilibrium given this mechanism, and for any realization of preference profiles, the equilibrium outcome maximizes welfare. We define asymptotic efficiency.

**Definition 2** For any $\rho \in \mathbb{R}^{++}$, a mechanism $c \in C$ is asymptotically $\rho$-efficient if for any $(F, \{\bar{w}_n\}_{n=1}^\infty, \gamma, p, G) \in \mathcal{F} \times \mathbb{R}^{\infty}_+ \times \mathbb{R}^{++} \times \mathbb{R}^{++} \times \mathcal{G}$, for any $\varepsilon \in (0, 1)$, and for any sequence $\{s^n\}_{n=2}^\infty$ such that $s^n \in E^{(n,F,\gamma,p,c,G)}$ for each $n \in \mathbb{N}\setminus\{1\}$, there exists $n_{\varepsilon,\gamma,F,p} \in \mathbb{N}$ such that for any $n > n_{\varepsilon,\gamma,F,p}$, the probability that the equilibrium outcome maximizes welfare $W_\rho$ converges to one.

That is, a mechanism is asymptotically $\rho$-efficient if as any society gets large, in any equilibrium, the mechanism maximizes welfare $W_\rho$ with probability converging to one.\(^6\) We

\(^5\)Moulin (1988) characterizes the set of Bergson welfare functions restricted to a domain of positive valuations. It is only a small corollary to identify which subset of the original characterization can be extended to a domain with negative valuations

\(^6\)Note that this definition is stricter than the usual definition of ex-post efficiency in mechanism design,
stress that a mechanism must asymptotically maximize welfare for any realization of society characteristics \((F, \{\bar{w}_n\}_{n=1}^{\infty}, \gamma, p, G)\) in order to be labeled “asymptotically efficient.”

Without further ado, we can anticipate a first result. For any Bergsonian welfare \(W_\rho\), we find an asymptotically \(\rho\)-efficient mechanism.

**Proposition 1** For any \(\rho \in \mathbb{R}\), the vote mechanism \(c\) defined by \(c(a) = |a|^{1+\frac{1}{\rho}}\) for any \(a \in \mathbb{R}\), is asymptotically \(\rho\)-efficient.

This result is obtained as a corollary of a general theorem below. We can reinterpret Proposition 1 in the language of implementation theory. The implementation question starts with a desired mapping from the realization of preferences for any society, to the subset of alternatives that are deemed desirable for this society and these preferences. This mapping is a social choice correspondence. A mechanism implements this social choice correspondence if all its equilibrium outcomes are in the social choice correspondence. So Proposition 1 says that for any positive real number \(\rho\), any sequence of social choice correspondences that consist of selecting the maximizer of a Bergson welfare function \(W_\rho\) is asymptotically implemented by a vote buying mechanism with cost function \(c(a) = |a|^{1+\frac{1}{\rho}}\).

We may wonder which other social choice correspondences are asymptotically implementable by vote-buying mechanisms. We next formalize this query, by defining the set of vote-buying mechanisms under consideration, and defining social choice correspondences and the notion of asymptotic implementability. We then characterize the set of social choice correspondences that are asymptotically implementable by vote-buying mechanisms.

**Admissible vote-buying mechanisms.** We restrict the collection of admissible cost functions to a set \(C\).

Let \(\hat{C}\) be the set of continuously differentiable functions defined over \(\mathbb{R}\) that are twice continuously differentiable over \(\mathbb{R}\setminus\{0\}\). Let \(C \equiv \{c \in \hat{C} : c(0) = 0, c'(0) = 0, \lim_{x \to 0} \frac{xc'(x)}{c(x)} \in \mathbb{R}\}\) where it suffices that there exists some equilibria that sustains efficient choices, and is more in the spirit of the implementation literature, which requires that all equilibria select a desirable outcome.
\[ (1, \infty), \ c'(x) > 0 \text{ for any } x' \in \mathbb{R}_+, \ \lim_{x \to \infty} c'(x) = \infty, \text{ and } c(x) = c(-x) \text{ for any } x \in \mathbb{R}, \} \]

be the set of admissible cost functions. The intuition for the conditions that we require all cost functions to satisfy, in addition to the continuity and differentiability assumptions, are as follows:

i) abstention (acquiring no votes) is free;

ii) to encourage positive participation, the marginal cost of votes at zero is zero, so for any strictly positive willingness to pay per vote, some strictly positive quantity of votes can be acquired at that price;

iii) but the elasticity of the cost function is greater than one (so \( c \) is convex) near zero, and thus the marginal cost of votes becomes immediately positive;

iv) and while elsewhere the cost function need not be convex, this marginal cost is always positive for all positive quantities;

v) and very high quantities of votes are prohibitively expensive; and

vi) neutrality: votes for \( A \) cost the same as votes against \( A \).

All power functions with exponent greater than one (and their sums), among other functions, are included in the set \( C \).

Social Choice correspondences. For any \( n \in \mathbb{N} \), a social choice correspondence \( SC^n : \mathbb{R}_+ \times \mathbb{R}_+ \times [-1, 1]^n \Rightarrow \{A, B\} \) maps a tuple \((\gamma, p, \theta_{N^n})\) into the subset of normatively desirable social decisions. Recall that for each agent \( i \in N^n \), the social choice is worth \( 2\gamma \theta_i \) units of real wealth to agent \( i \), or \( 2\frac{\gamma}{p} \theta_i \) monetary units. We refer to \( \frac{\gamma}{p} \theta_i \) as agent \( i \)’s valuation and to \( \frac{\gamma}{2} \theta_{N^n} \) as the valuation profile. We are interested only in social choice correspondences that depend only on the valuation profile, mapping the set of all possible valuation profiles, into the set of socially desirable decisions.

Let \( SC = \{SC^n\}_{n=1}^\infty \) denote a sequence of social choice correspondences. For any \( n \in \mathbb{N} \), and any social choice correspondence \( SC^n \), define

\[
\Theta_A(SC^n) \equiv \{(\gamma, p, \theta_{N^n}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times [-1, 1]^n : SC^n(\gamma, p, \theta_{N^n}) = A\}
\]
and
\[ \Theta_B(SC^n) \equiv \{ (\gamma, p, \theta_N^n) \in \mathbb{R}^+ \times \mathbb{R}^+ \times [-1, 1]^n : SC^n(\gamma, p, \theta_N^n) = B \}. \]

For each \( \rho \in \mathbb{R}^+ \), and for each \( n \in \mathbb{N} \), define the social choice correspondence \( SC^n_\rho \) by
\[ SC^n_\rho(\gamma, p, \theta_N^n) = \arg \max W(\theta_N^n). \] Let \( SC_\rho \equiv \{ SC^n_\rho \}_{n=1}^\infty \) and \( SC \equiv \bigcup_{\rho \in \mathbb{R}^+} SC_\rho. \)

For each \( n \in \mathbb{N} \) and \( J \in \{ A, B \} \), let \((\Theta_J(SC^n))^c \equiv [-1, 1]^n \setminus \Theta_J(SC^n)\) denote the complement of \( \Theta_J(SC^n) \). For any \( n \in \mathbb{N} \) and any subset of \( \mathbb{R}^n \), let \( L \) denote the Lebesgue measure over \( \mathbb{R}^n \).

**Definition 3** A pair of sequences of social choice correspondences \( SC \) and \( \tilde{SC} \) converge to each other almost everywhere if, for any \((\gamma, p) \in \mathbb{R}^2_+\)
\[ \lim_{n \to \infty} \frac{L(\Theta_J(SC^n) \cap (\Theta_J(\tilde{SC}^n))^c)}{2^n} = 0 \]
for each \( J \in \{ A, B \} \).

That is, the two sequences of social choice correspondences converge to each other almost everywhere if the set of valuations for which they select different social decisions becomes vanishingly small as society gets arbitrarily large.

**Implementability.**

**Definition 4** A vote-buying mechanism \( c \in C \) asymptotically implements a sequence of social choice correspondences \( SC \) in symmetric, pure, neutral and monotone equilibria if for any \((F, \{ \bar{w}_n \}_{n=1}^\infty, \gamma, p, G) \in \mathcal{F} \times \mathbb{R}^\infty \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{G}, \) for any \( \varepsilon \in (0, 1) \), and for any sequence \( \{ s^n \}_{n=2}^\infty \) such that \( s^n \in E^{(n, F, \gamma, p, c, G)} \) for each \( n \in \mathbb{N} \setminus \{1\} \), there exists \( n_{\varepsilon, \gamma, F, p, G} \in \mathbb{N} \) such that
for any \( n > n_{\varepsilon, F, p, G} \),

\[
\frac{2^n}{L(\Theta_A(SC^n))} \int_{(\gamma, p, \theta_{N^n}) \in \Theta_A(SC^n)} \left( \prod_{i=1}^{n} f(\theta_i) \right) G \left( \sum_{i=1}^{n} s^n(\theta_i) \right) d\theta_{N^n} > 1 - \varepsilon \text{ and }
\frac{2^n}{L(\Theta_B(SC^n))} \int_{(\gamma, p, \theta_{N^n}) \in \Theta_B(SC^n)} \left( \prod_{i=1}^{n} f(\theta_i) \right) G \left( \sum_{i=1}^{n} s^n(\theta_i) \right) d\theta_{N^n} < \varepsilon.
\]

We say that a sequence of social choice correspondences \( SC \) is asymptotically implementable if there exists a mechanism \( c \in C \) that asymptotically implements \( SC \) in symmetric, pure, neutral and monotone equilibria.

Note that for each \( J \in \{A, B\}, \left( \prod_{i=1}^{n} f(\theta_i) \right) \) is the probability that \( \theta_N \) is realized, and

\[
\frac{L(\Theta_J(SC^n))}{2^n} \int_{(\gamma, p, \theta_{N^n}) \in \Theta_J(SC^n)} \left( \prod_{i=1}^{n} f(\theta_i) \right) G \left( \sum_{i=1}^{n} s^n(\theta_i) \right) d\theta_{N^n}
\]

is just the density of \( \theta_{N^n} \) conditional on \( (\gamma, p, \theta_{N^n}) \in \Theta_J(SC^n) \).

3 Main Result

We provide two intermediate results of interest, leading to our main result: a characterization of the set of sequences of social choice correspondences that are asymptotically implementable by vote-buying mechanisms, together with, for each sequence of social choice correspondences that is implementable, a class of vote-buying mechanisms that asymptotically implements it.

For each \( n \in \mathbb{N} \setminus \{1\} \), for any voter \( i \in N^n \), for any type \( \theta_i \in [-1, 1] \), and for any strategy profile \( s_{-i} \) for the other players, \( a_i \notin [-c^{-1}(2\gamma), c^{-1}(2\gamma)] \) is a dominated action: it leads to a strictly lower payoff than \( a_i = 0 \). Thus, we can restrict attention to a restricted game with bounded action space \( \bar{A} \equiv [-c^{-1}(2\gamma), c^{-1}(2\gamma)] \). For each tuple \( (n, F, \gamma, \bar{w}_{N^n}, p, c, G) \in \mathbb{N} \setminus \{1\} \times \mathcal{F} \times \mathbb{R}^{++} \times \mathbb{R}^{++} \times C \times \mathcal{G} \), define \( S^\gamma \) as the set of all functions \( y : [-1, 1] \rightarrow [-c^{-1}(2\gamma), c^{-1}(2\gamma)] \), and let \( \Gamma^{(n, F, \gamma, \bar{w}_{N^n}, p, c, G)} \) denote the restricted game played by the \( n \) players in society \( N^n \), with strategy set \( S^\gamma \) for each agent, and expected utility given by \( EU_i \) in expression 1 for each \( i \in N^n \). We first establish an existence result.
Lemma 2  For any tuple \((n, F, \gamma, \bar{w}_{N^n}, p, c, G)\) \(\in \mathbb{N}\setminus\{1\} \times \mathcal{F} \times \mathbb{R}_{++} \times \mathbb{R}_{++}^n \times \mathbb{R}_{++} \times C \times G\), a symmetric pure neutral monotone equilibrium of game \(\bar{\Gamma}^{(n, F, \gamma, \bar{w}_{N^n}, p, c, G)}\) exists.

Since any equilibrium of \(\bar{\Gamma}^{(n, F, \gamma, \bar{w}_{N^n}, p, c, G)}\) is also an equilibrium of \(\Gamma^{(n, F, \gamma, \bar{w}_{N^n}, p, c, G)}\), it follows as a corollary that a symmetric pure neutral monotone equilibrium of game \(\Gamma^{(n, F, \gamma, \bar{w}_{N^n}, p, c, G)}\) exists as well. Equilibrium strategies converge to zero as society becomes large (Lemma 4 in the Appendix). Equilibrium strategies also converge to linear in a given power of individual valuations (Lemma 10). We use these asymptotic properties to characterize the set of sequences of social choice correspondences that are asymptotically implementable by a vote-buying mechanism.

Theorem 3  A sequence \(SC\) of social choice correspondences is asymptotically implementable by a vote-buying mechanism in \(C\) if and only if there exists \(\rho \in (0, \infty)\) such that \(SC\) and \(SC_{\rho}\) converge to each other almost everywhere, in which case, any vote-buying mechanism \(c \in C\) such that \(\lim_{x \to 0^+} \frac{xc'(x)}{c(x)} = \frac{1+\rho}{\rho}\) asymptotically implements \(SC\).

That is, only sequences of social choice correspondences that converge toward selecting the Bergson \(W_{\rho}\) maximizer for some \(\rho \in \mathbb{R}_{++}\) are asymptotically implementable by vote buying mechanisms. Note as a corollary that a mechanism \(c\) with \(c(x) = |x|^{\frac{1+\rho}{\rho}}\) satisfies \(\lim_{x \to 0^+} \frac{xc'(x)}{c(x)} = \frac{1+\rho}{\rho}\) and thus it implements \(SC_{\rho}\) (Proposition 1).

4 Discussion

Given a binary collective choice problem, a social choice correspondence is efficient if it maximizes aggregate welfare. Aggregate welfare can be defined in many ways, depending on our normative criterion to evaluate trade-offs between pleasing more agents, or pleasing agents with more intense preferences.

A particular class of Bergson welfare functions, characterized by a set of normatively appealing axioms, is indexed by a parameter \(\rho\) that measures how much we care about intensity.
At one end of this class, majority rule assigns equal importance to each individual ordinal preference, entirely disregarding intensity. At the opposite end of the class, a notion that cares maximally about intensity equates welfare with the utility of the agent with the most intense preference. Utilitarianism is an interior principle, caring for all agents’ preferences in linear proportion to their intensity.

For any welfare criterion in this axiomatized class, we find a vote-buying mechanism that is asymptotically \( \rho \)-efficient, that is, that is efficient according to the welfare function given by the desired \( \rho \). In particular, for each welfare criterion with attention to intensity \( \rho \), a mechanism with cost of votes proportional to \( |x|^{1+\rho} \) is asymptotically efficient according to this welfare notion.

We further obtain a complete characterization of the set of social choice correspondences that are asymptotically implementable: a sequence of social choice correspondences is asymptotically implementable if and only if it converges to almost always making efficient choices according to a Bergson welfare function \( W_\rho \) for some positive \( \rho \) (Theorem 3).

The standard relative majority voting rule that assigns one vote to each person for free is equivalent to the limit point \( \rho = 0 \) of our range of mechanisms: as the cost \( c(x) = |x|^{1+\rho} \) of votes diverges to \( \lim_{\rho \to 0} |x|^{1+\rho} = |x|^{\infty} \), any real number of votes up to one becomes free, and any number beyond one becomes too expensive, so everyone converges toward acquiring one vote.

A decentralized, competitive market for votes, similar to the ones proposed for instance by Dekel, Jackson and Wolinski (2008) and Casella, Llorente-Saguer and Palfrey (2012), implements the opposite extreme, \( \rho = \infty \): as the cost converges to \( \lim_{\rho \to \infty} |x|^{1+\rho} = |x| \), the pricing of votes becomes linear, as in a competitive market, and the agent or agents with most intense preferences purchase most votes and determine the social decision.

Casella, Llorente-Saguer and Palfrey (2012) interpret the outcome with a market for votes as a social welfare loss, because they judge welfare according to an utilitarian perspective. We interpret the finding differently: the outcome is optimal according to a welfare notion in which we care overwhelmingly more about the agent with the most intense preference. For
that welfare criterion, a market for votes with linear pricing, be it a centralized one as in our mechanism, or a decentralized one as in Casella, Llorente-Saguer and Palfrey (2012), is optimal. If that is not the welfare criterion we have in mind, then we should not choose linear pricing for votes. Rather, we should choose the pricing that corresponds to our welfare notion. For utilitarian welfare, corresponding to a parameter value of $\rho = 1$, quadratic pricing is optimal (Lalley and Weyl 2016). For any other welfare notion corresponding to parameter value $\rho \in \mathbb{R}_{++}$, an optimal pricing of votes is $c(x) = |x|^{\frac{1+\rho}{\rho}}$.

We address three limitations.

**Wealth inequality.**

A common criticism of vote-buying mechanisms that rely on linear or quadratic pricing is that in practice they would favor the rich, effectively disenfranchising the poor. In our theory, as in previous theories of vote-buying mechanisms, agents are risk neutral and preferences over wealth are separable, so the utility representation is quasilinear and there are no wealth effects: agents’ actions are independent of their wealth.

Concerns about the effects of wealth inequality arise if we assume that agents are risk averse, so that their utility over wealth is concave. If so, for any given preference intensity over the social choice, a wealthier agent would acquire more votes than an agent with the same intensity of preference and lesser wealth. If the planner cares only about the social decision, and not about wealth redistribution, the efficiency of the mechanisms we have studied is lost: since the cost function conditions only on the number of votes, the preferences of wealthier votes are overweighed. Efficiency can be restored by allowing for vote-buying mechanisms such that the cost function conditions on wealth and on the number of votes (this result is available from the authors).

**Multiple alternatives.**

We have identified mechanisms to make binary social decision. If the set of alternatives
under consideration contains multiple alternatives, the efficiency properties of these vote-
buying mechanisms are weakened. As in elections with multiple candidates, coordination can
result in only two alternatives being competitive, so agents purchase and cast votes for only
these two. These two alternatives may be any pair, and not necessarily the two most efficient.
Our result, in this case, only implies that the least efficient alternative will be defeated with
probability converging to one. This limitation is not intrinsic to vote-buying mechanism; it is
a feature shared by standard voting practices in which each agent has one vote.

Non-neutral distribution of preferences.

We have assumed that the distribution of preferences in any society is neutral over the two
alternatives: the mean and median valuation is zero. Extending our results to distributions
that are non-neutral, so that the mean and median favors one alternative over another, is the
subject of our ongoing research agenda.

We have shown that binary social choice correspondences that choose the alternative that
maximizes a Bergson social welfare function with parameter \( \rho \), can be asymptotically imple-
mented via a vote-buying mechanism with any cost function whose elasticity converges to \( \frac{1+\rho}{\rho} \)
at zero votes.

5 Appendix

5.1 Assumptions

Recall that for any \( n \in \mathbb{N}\backslash\{1\} \) and for any \( i \in N^n \), \( \succsim_i \) is a complete and transitive relation
over the set of probability measures \( \mathcal{M}((A, B) \times \mathbb{R}^n) \). For any \( n \in \mathbb{N}\backslash\{1\} \), and for any \( i \in N^n \),
we assume the following on \( \succsim_i \).

**Assumption 2.** The reference relation \( \succsim_i \) is continuous and satisfies independence over
decomposition of lotteries.
From Assumption 2, it follows that each preference relation \( \succsim_{Nn} \) is representable by a continuous utility function in expected utility form (von Neumann and Morgenstern’s (1944) Expected Utility theorem).

**Assumption 3.** For any \( (d, \frac{w_{Nn}}{p}) \in \{A, B\} \times \mathbb{R}^n \) and \( (d, \frac{w_{Nn}'}{p}) \in \{A, B\} \times \mathbb{R}^n \) such that \( w_i = w_i' \), agent \( i \) is indifferent between \( (d, \frac{w_{Nn}}{p}) \) and \( (d, \frac{w_{Nn}'}{p}) \).

Assumption 3 means that each agent cares only about the social choice, and about her own real final wealth. With a slight abuse of notation we can then refer \( \succsim_i \) to a preference order over \( \Delta(\{A, B\} \times \mathbb{R}) \).

**Assumption 4.** For any \( (d, \frac{w_{Nn}}{p}) \in \{A, B\} \times \mathbb{R}^n \) and \( (d, \frac{w_{Nn}'}{p}) \in \{A, B\} \times \mathbb{R}^n \) such that \( w_i > w_i' \), agent \( i \) strictly prefers \( (d, \frac{w_{Nn}}{p}) \) to \( (d, \frac{w_{Nn}'}{p}) \).

Assumption 4 means that each agent has strictly monotonically increasing preferences over final real wealth.

Let a 50-50 lottery be a probability distribution over \( \{A, B\} \times \mathbb{R}^n \) that assigns probability 0.5 to exactly two outcomes. For any lottery \( L \in \mathcal{M}(\{A, B\} \times \mathbb{R}^n) \), let \( L_d \) and \( L_{w_i/p} \) be the marginal distribution over \( d \) and over \( w_i/p \) (respectively) given \( L \).

**Assumption 5.** (Fishburn [13] Separability) For any two 50-50 lotteries \( L, L' \) such that \( L_d = L'_d \) and \( L_{w_i/p} = L'_{w_i/p} \), \( L \sim_i L' \).

This means that preferences over lotteries are driven only by the marginal probability distributions, and not by their correlation. This is a separability condition, because it implies that the preferences over lotteries in one dimension do not change with changes in the other dimension.

Assumptions 1-3 and 5 jointly imply that the preferences of agent \( i \) can be represented by an additively separable function of the outcome and the final real wealth of \( i \) in expected utility form (Fishburn [13], Theorem 11.1).

**Assumption 6.** Agent \( i \) is risk neutral with respect to final real wealth.

Given any probability measure \( \mu \in \mathcal{M}(\{A, B\} \times \mathbb{R}^n) \), let \( \mu_d \) and \( \mu_{w_i/p} \) be the associated marginal probability measures over \( \{A, B\} \) and over \( \mathbb{R} \), derived from \( \mu \). Note that for each
\( x \in \{A, B\}, \mu_d(\{x\}) = \Pr[d = x], \) while for each \( x \in \mathbb{R}, \mu_{w_i/p}(\{x\}) = \Pr \left[ \frac{w_i}{p} = x \right] \). For any interval \([x, \frac{w_i}{p}] \subseteq \mathbb{R}, \) let \( \mu'_{w_i/p} \left( [x, \frac{w_i}{p}] \right) = \frac{\delta}{\delta(w_i/p)} \mu_{w_i/p} \left( [x, \frac{w_i}{p}] \right) \) be the density function associated to the marginal probability measure \( \mu_{w_i/p} \). Hence, for any interval \( I \subset \mathbb{R}, \)

\[
\Pr \left[ \frac{w_i}{p} \in I \right] = \sum_{\frac{w_i}{p} \in I : \mu(\{\frac{w_i}{p}\}) > 0} \mu \left( \left\{ \frac{w_i}{p} \right\} \right) + \int_{x \in I} \mu'_{w_i/p}(\{x\}) \, dx,
\]

where the first term captures the probability mass points, and the second the integral over the density, wherever defined.

Assumptions 1-6 jointly imply that \( \succsim_i \) is representable by an additively separable, quasi-linear utility function \( \tilde{u}_i \) such that for each \( \mu \in \mathcal{M}(\{A, B\} \times \mathbb{R}^n), \)

\[
\tilde{u}_i(\mu) = \sum_{d \in \{A, B\}} \mu_d(\{x\}) u_d^i(x) + \sum_{w_i \in \mathbb{R}} \mu_{w_i/p} \left( \left\{ \frac{w_i}{p} \right\} \right) \frac{w_i}{p} + \int_{x \in \mathbb{R}} \mu'_{w_i/p}(\{x\}) \, dx,
\]

where \( u_d^i : \{A, B\} \to \mathbb{R} \) is a function that represents the preferences over the social choice.

### 5.2 Proofs

**Proof of Lemma 2.** Consider a further restricted game in which each player \( i \) privately observes \(|\theta_i|\) but not \( \theta_i \) and then chooses an action \(|a_i| \in \mathbb{R}_+\). Subsequently, agent \( i \) learns the sign of \( \theta_i \) and chooses the sign of \( a_i \). Since it is dominated to choose \( \text{sgn}(a_i) \neq \text{sgn}(\theta_i) \), assume that \( \text{sgn}(a_i) = \text{sgn}(\theta_i) \). Let \( \hat{S}^\gamma \) as the set of all functions \( \tilde{s} : [0, 1] \to [0, c^{-1}(2\gamma)] \), and let \( \hat{\Gamma}^{(n,F,\gamma,\tilde{a},N^\gamma,p,c,G)} \) denote the restricted game played by the \( n \) players in society \( N^n \), with strategy set \( \hat{S}^\gamma \) for each agent, and expected utility given by \( EU_i \) in Expression (1) for each \( i \in N^n \).

Note that game \( \hat{\Gamma}^{(n,F,\gamma,\tilde{a},N^\gamma,p,c,G)} \) satisfies the nine conditions for existence of a symmetric, pure monotone equilibrium in Reny’s (2011) Theorem 4.5. Conditions G1-G6 in this theorem, as explained by Reny, are standard and applied to a vast class of more general environments that includes our own as a very special case. The three additional conditions are the following:
i) the game must be symmetric. Game $\hat{G}^{(n,F,\gamma,\bar{w}_{N^n},p,c,G)}$ is symmetric, because each player’s preference is drawn from the same distribution $F$, and $G$ is anonymous, aggregating total contributions.

ii) each player’s set of monotone pure best replies is non-empty. Given the actions by other players, each player $i$ maximizes a continuous function over a compact domain, so a maximum exists, and this maximum is a best response. Furthermore, the utility function is supermodular in $|\theta_i|$ and $|a_i|$ (it satisfies increasing differences in $(|\theta_i|, |a_i|)$ and so the set of maximizers is non-decreasing, and thus we can select a monotonically increasing best response.

iii) each player’s set of monotone pure best replies is join-closed whenever the other players employ the same monotone pure strategy. A subset of strategies is join-closed if the pointwise supremum of any pair of strategies in the set is also in the set. Since the maximization problem is independently solved for each $|\theta_i|$ to obtain a best response, the pointwise maximum of any pair of strategies is in the set. Since the set of best responses is closed, the pointwise supremum is a pointwise maximum, for each point.

Therefore, game $\hat{G}^{(n,F,\gamma,\bar{w}_{N^n},p,c,G)}$ has a symmetric, pure monotone equilibrium. This equilibrium is neutral by construction. We next show that this equilibrium is also an equilibrium of the game $\bar{G}^{(n,F,\gamma,\bar{w}_{N^n},p,c,G)}$. Denote $s^n$ the strategy played in a symmetric neutral, monotone, pure equilibrium of game $\bar{G}^{(n,F,\gamma,\bar{w}_{N^n},p,c,G)}$, and assume that $s^n$ is not an equilibrium strategy of $\bar{G}^{(n,F,\gamma,\bar{w}_{N^n},p,c,G)}$. Then, there exists $\theta$ such that any agent $i$ with $\theta_i = \theta$ prefers to deviate to $s_i = s'$ with $s'(\theta) \neq s^n(\theta)$. Since $s^n$ is neutral and $G(x) - \frac{1}{2} = \frac{1}{2} - G(-x)$, the utility for an agent $j$ with $\theta_j = -\theta$ of deviating to play $a_j = -s'(\theta)$ equals the utility for $i$ of deviating to play $a_i = -s'(\theta)$, and thus $j$ would deviate as well. But then, $s^n(|\theta|) = |s^n(\theta)|$ is not a best response in game $\bar{G}^{(n,F,\gamma,\bar{w}_{N^n},p,c,G)}$, since for $|\theta|$, any agent $i$ prefers to deviate to $|s'(\theta)|$. So we arrive at a contradiction. It must thus be that $s^n$ is also an equilibrium of $\bar{G}^{(n,F,\gamma,\bar{w}_{N^n},p,c,G)}$.

For each $n \in \mathbb{N}\setminus\{1\}$, let $\{\bar{\theta}_1, ..., \bar{\theta}_n\}$ be $n$ independent random variables with cumulative distribution $F$, and denote by $H^n$ the cumulative distribution function of the random variable $\sum_{j \in N \setminus \{i\}} s^n(\bar{\theta}_j)$; by equilibrium symmetry, $H^n$ does not depend on $i \in N^n$. 

22
Notice that since it is strictly dominated for any player with valuation zero to incur costs, it follows that \( s^n(0) = 0 \) for any \( n \in \mathbb{N} \setminus \{1\} \). Further, since \( s^n \) is monotonic and neutral and the equilibrium is symmetric, \( s(1) > 0 \), because if \( s(1) = 0 \), then \( s(\theta) = 0 \) for any \( \theta \in [-1, 1] \), and if so, any agent with \( \theta_i \neq 0 \) prefers to deviate to invest a positive quantity. Therefore, the variance of \( H^n \) is strictly positive.

For any \( n \in \mathbb{N} \setminus \{1\} \), any strategy \( s^n \in E^{(n,F;\gamma,p,c,G)} \) is such that no agent chooses an action \( a_i \in \mathbb{R} \) with \( |a_i| > c^{-1}(2\gamma) \), because doing so costs more than \( 2\gamma \), which is the maximum utility that any individual can derive from the social choice. Therefore, \( H^n(x) = 0 \) for any \( x < -(n-1)c^{-1}(2\gamma) \) and \( H((n-1)c^{-1}(2\gamma)) = 1 \).

**Lemma 4** For any \( n \in \mathbb{N} \setminus \{1\} \), and for any \( s^n \in E^{(n,F;\gamma,p,c,G)} \), \( s^n \) is strictly increasing.

**Proof.** Recall \( \bar{A} \equiv [-c^{-1}(2\gamma), c^{-1}(2\gamma)] \), and for any \( n \in \mathbb{N} \setminus \{1\} \) and any \( x \in (n-1)\bar{A} \), define \( \psi^n(x) \equiv \Pr[\sum_{j \in N \setminus \{i\}} s^n(\theta_j) = x] \), and define \( h^n : (n-1)\bar{A} \) as the probability density over \( H^n \) such that

\[
\sum_{x \in (n-1)\bar{A}} \psi^n(x) + \int_{-(n-1)\bar{A}}^{(n-1)\bar{A}} h^n(x) dx = 1.
\]

Then, given any equilibrium \( s^n \in E^{(n,F;\gamma,p,c,G)} \), the optimization problem of player \( i \) with valuation \( \theta_i \in [-1, 1] \) is

\[
\max_{a_i \in \bar{A}} \gamma \theta_i \left( \sum_{x \in (n-1)\bar{A}} \psi^n(x)G(x + a_i) + \int_{-(n-1)\bar{A}}^{(n-1)\bar{A}} h^n(x)G(x + a_i) dx \right) - G(x + a_i) + \int_{-(n-1)\bar{A}}^{(n-1)\bar{A}} h^n(x)(1 - G(x + a_i)) dx - c(a_i),
\]

or equivalently,

\[
\max_{a_i \in \bar{A}} \gamma \theta_i \left( \sum_{x \in (n-1)\bar{A}} \psi^n(x)(2G(x + a_i) - 1) + \int_{-(n-1)\bar{A}}^{(n-1)\bar{A}} h^n(x)(2G(x + a_i) - 1) dx \right) - c(a_i).
\]
Since $G$ is continuously differentiable and the constraint $a_i \in \bar{A}$ is not binding, we obtain the solution by the First Order Condition

$$2\gamma_{t_i} \left( \sum_{x \in (n-1)\bar{A}} \psi^n(x)g(x + a_i) + \int_{-\frac{x}{(n-1)\bar{A}}}^{\frac{(n-1)\bar{A}}{x}} g(x + a_i)h^n(x)dx \right) = \gamma'(a_i). \tag{2}$$

Note that since $g$ is strictly positive in $\mathbb{R}$, and \( \sum_{x \in (n-1)\bar{A}} \psi^n(x) + \int_{-\frac{x}{(n-1)\bar{A}}}^{\frac{(n-1)\bar{A}}{x}} h^n(x) = 1 \), it follows that the left hand side of Equation (2) is non-negative for any $a_i \in \bar{A}$ and any $\theta_i \in [-1, 1]$, and it is strictly increasing in $\theta_i$.

Assume $a_k$ is a solution to the First Order Condition (2) for $i = k$, and assume $\theta_j > \theta_k$ and $a_j = a_k$. Then $a_j$ does not solve the First Order Condition (2) for $i = j$, because the right hand side is equal for $k$ and $j$, but the left hand side is strictly greater for $j$ than for $k$. Thus, it follows that for any two distinct $\theta, \theta'$, we obtain $s^n(\theta) \neq s^n(\theta')$, which, since $s^n$ is weakly increasing, implies that it is strictly increasing.

As an immediate corollary of Lemma 4, $H^n$ does not have a mass point, for each $n \in \mathbb{N} \setminus \{1\}$ we can define the probability density function $h^n : (n - 1)\bar{A} \to \mathbb{R}_+$ such that \( \int_{-\frac{x}{(n-1)\bar{A}}}^{\frac{x}{(n-1)\bar{A}}} h^n(t)dt = H^n(x) \). Given any equilibrium $s^n \in E^{(n,F;\gamma;p,c,G)}$, the first order condition of the maximization problem of each agent $i \in N^n$ can be simplified to

$$2\gamma_{t_i} \int_{-\frac{x}{(n-1)\bar{A}}}^{\frac{(n-1)\bar{A}}{x}} g(x + a_i)h^n(x)dx = \gamma'(a_i). \tag{3}$$

Lemma 5 establishes that vote acquisitions converge to zero.

**Lemma 5** For any tuple $(F, \gamma, \{\hat{w}_n\}_{n=1}^\infty, p, c, G) \in \mathbb{N} \setminus \{1\} \times \mathcal{F} \times \mathbb{R}_+^+ \times \mathbb{R}_+^\infty \times \mathbb{R}_+^+ \times C \times \mathcal{G}$, and any sequence $\{s^n\}_{n=1}^\infty$ such that $s^n \in E^{(n,F;\gamma;p,c,G)}$ for each $n \in \mathbb{N} \setminus \{1\}$, \( \lim_{n \to +\infty} s^n(\theta) = 0 \) for each $\theta \in (-1, 1)$.

**Proof of Lemma 5.** Proof by contradiction. For any tuple $(F, \gamma, \{\hat{w}_n\}_{n=1}^\infty, p, c, G) \in \mathbb{N} \setminus \{1\} \times \mathcal{F} \times \mathbb{R}_+^+ \times \mathbb{R}_+^\infty \times \mathbb{R}_+^+ \times C \times \mathcal{G}$, assume that $\{s^n\}_{n=2}^\infty$ is a sequence of neutral, monotone,
symmetric, pure equilibrium strategies of game $\Gamma^{(n,F,\gamma,\xi,N,\psi,c,G)}$, and assume (absurd) that there exists $\theta' \in (-1,1)$ such that $\lim_{n \to +\infty} s^n(\theta') \neq 0$.

Then there exist a $\delta \in \mathbb{R}_{++}$ and an infinite subsequence $\{s^n(\tau)\}_{\tau=1}^\infty$ of $\{s^n\}_{n=2}^\infty$ with $n : \mathbb{N}\{1\} \to \mathbb{N}$ strictly increasing, such that $|s^n(\tau)(\theta')| \geq \delta$ for every $\tau \in \mathbb{N}$. Note $n(\tau)$ is the size of the society in the $\tau$-th element of the subsequence. By monotonicity of $s^n(\tau)(\theta)$ with respect to $\theta \in [-1,1]$ for each $\tau \in \mathbb{N}$, it follows that if $\theta' \in (-1,0)$, then $s^n(\tau)(\theta) \leq -\delta$ for any $\theta \in [-1,\theta']$ and for any $\tau \in \mathbb{N}$, and if $\theta' \in (0,1)$, then $s^n(\tau)(\theta) \geq \delta$ for any $\theta \in [\theta',1]$.

For each $n \in \mathbb{N}\{1\}$ and for each $k \in \{1,\ldots,n\}$, define the random variable $s^n_k(\bar{\theta}) \equiv s^n(\bar{\theta})$. These are $n$ independent, identically distributed random variables. For each $\tau \in \mathbb{N}$, and for each $k \in \{1,\ldots,n(\tau)\}$, let $E[s_k^n(\bar{\theta})]$ denote the expectation of the random variable $s_k^n(\bar{\theta})$; by neutrality of $s^n$, $E[s_k^n(\bar{\theta})] = 0$. For each $\tau \in \mathbb{N}$, and for each $k \in \{1,\ldots,n(\tau)\}$, the variance of $s_k^n(\bar{\theta})$ is $\sigma^2 = E[s_k^n(\bar{\theta})]^2 = E[s_k^n(\bar{\theta})^2] = E[s_k^n(\bar{\theta})]E[s_k^n(\bar{\theta})] = E[s_k^n(\bar{\theta})^2]$. By monotonicity of $s^n(\tau)$, $E[s_k^n(\bar{\theta})^2] \geq 2(1-F(\theta'))\delta^2 > 0$. Further, note that as $s^n(\tau)$ is bounded, the third moment of $s_k^n(\bar{\theta})$, denoted as $E[s_k^n(\bar{\theta})^3]$, also bounded; further, since $E[s_k^n(\bar{\theta})^2] \geq 2(1-F(\theta'))\delta^2$, it follows $E[s_k^n(\bar{\theta})^3] \geq (2-2F(\theta'))\frac{3}{2}\delta^3 > 0$. For each $\tau \in \mathbb{N}$, define $V^\tau$ as the cumulative distribution of the random variable $\sum_{k \in N(\tau)} \bar{s}_k$, denoted as $N[\sum_{k \in N(\tau)} s_k^n(\bar{\theta})]$. For any $z \in \mathbb{R}_{++}$, let $N[0,z]$ denote a normal distribution with zero mean and variance $z$, and for any $x \in \mathbb{R}$, let $N[0,z](x)$ the value of the cumulative distribution of $N[0,z]$ at $x$.

By the Berry–Esseen theorem (Berry 1941, Esseen 1942), there exists $\kappa \in \mathbb{R}_{++}$ such that for any $\tau \in \mathbb{N}$ and any $x \in \mathbb{R}$,

$$|V^\tau(x) - N[0,1](x)| < \frac{\kappa E[s_k^n(\bar{\theta})^3]}{\sqrt{n(\tau) - 1} (E[s_k^n(\bar{\theta})^2])^{\frac{3}{2}}} ,$$

or equivalently,

$$|H^\tau(x) - N[0,E[s_k^n(\bar{\theta})^2](n(\tau) - 1)](x)| < \frac{\kappa E[s_k^n(\bar{\theta})^3]}{\sqrt{n(\tau) - 1} (E[s_k^n(\bar{\theta})^2])^{\frac{3}{2}}} , \quad (4)$$

25
which implies that
\[ |H^\tau(x) - N[0, E[|s^{n(\tau)}(\tilde{\theta})|^2] (n(\tau) - 1)](x)| < \frac{\kappa E[|s^{n(\tau)}(\tilde{\theta})|^3]}{\sqrt{n(\tau) - 1}(2 - 2F(\theta'))}\frac{3}{2}. \quad (5) \]

Since \(\{s^{n(\tau)}(\tilde{\theta})\}_{\tau=1}^\infty\) is a bounded sequence, \(\{E[|s^{n(\tau)}(\tilde{\theta})|^3]\}_{\tau=1}^\infty\) is bounded as well, and the right hand side of Inequality (5) converges to zero as \(\tau\) diverges toward \(\infty\), and thus, \(H^\tau\) converges as \(\tau \to \infty\) to a normal distribution with mean zero and variance \((n(\tau) - 1)E[|s^{n(\tau)}(\tilde{\theta})|^2]\). Since \(E[|s^{n(\tau)}(\tilde{\theta})|^2] \geq 2(1 - F(\theta'))\delta^2\) and \(n(\tau)\) diverges toward \(\infty\) in \(\tau\), it follows that the variance \((n(\tau) - 1)E[|s^{n(\tau)}(\tilde{\theta})|^2]\) diverges to infinity and the probability that \(\sum_{k \in \mathbb{N} \setminus \{i\}} s^{n(\tau)}(\tilde{\theta}_j)\) belongs to \([-x, x]\) converges to zero for any \(x \in \mathbb{R}\). That is, for any \(x \in \mathbb{R}_{++},\)
\[
\lim_{\tau \to \infty} (H^\tau(x) - H^\tau(-x)) = 0. \quad (6)
\]

Since \(g\) is strictly increasing, neutral \((G(x) = 1 - G(-x))\) and \(\lim_{x \to -\infty} G(x) = 0\), then for any \(\varepsilon \in (0, \frac{1}{2}c(\delta))\), there exist \(x \in \mathbb{R}_{++}\) such that for any \(x \in (-\infty, -\tilde{x}] \cup [\tilde{x}, \infty),\)
\[
|G(x + c^{-1}(2\gamma)) - G(x)|2\gamma\theta' < \frac{1}{2}c(\delta) - \varepsilon.
\]
Since \(|s^{n(\tau)}(\theta')| \geq \delta\) for every \(\tau \in \mathbb{N}\) and \(c\) is increasing, it follows that
\[
|G(x + c^{-1}(2\gamma)) - G(x)|2\gamma\theta' < c(s^{n(\tau)}(\theta')) - \varepsilon
\]
for any \(x \in (-\infty, -\tilde{x}] \cup [x, \infty).\) Further, since \(|s^{n(\tau)}(\theta')| \leq c^{-1}(2\gamma)\) (because \(|s^{n(\tau)}(\theta')| > c^{-1}(2\gamma)\) implies that \(s_i = s^{n(\tau)}\) is a strictly dominated strategy), it follows that for any \(x \in (-\infty, -\tilde{x}] \cup [\tilde{x}, \infty),\)
\[
|G(x + s^{n(\tau)}(\theta')) - G(x)|2\gamma\theta' < c(s^{n(\tau)}(\theta')) - \varepsilon. \quad (7)
\]

For each \(\tau \in \mathbb{N}\), for an arbitrary agent \(i\), the expected utility of playing \(s_i = s^{n(\tau)}\), minus
the expected utility of playing \( a_i = 0 \), given that \( \theta_i = \theta' \), is:

\[
2\gamma\theta' \int_{-\infty}^{\infty} \left( G(x + s^{n(\tau)}(\theta')) - G(x) \right) h^\tau(x) dx + 2\gamma\theta' \int_{-\infty}^{\infty} \left( G(x + s^{n(\tau)}(\theta')) - G(x) \right) h^\tau(x) dx \\
+ 2\gamma\theta' \int_{-\infty}^{\infty} \left( G(x + a_i) - G(x) \right) h^\tau(x) dx - c(s^{n(\tau)}(\theta')),
\]

(8)

From Equality 6, \( \lim_{\tau \to \infty} h^{n(\tau)}(x) = 0 \) for any \( x \in [-c^{-1}(2\gamma), c^{-1}(2\gamma)] \), and hence

\[
\lim_{\tau \to \infty} 2\gamma\theta' \int_{-\infty}^{\infty} \left( G(x + s^{n(\tau)}(\theta')) - G(x) \right) h^\tau(x) dx = 0.
\]

Therefore, the limit of Expression (8) as \( \tau \to \infty \) is equal to the limit of

\[
2\gamma\theta' \int_{-\infty}^{\infty} \left( G(x + s^{n(\tau)}(\theta')) - G(x) \right) h^\tau(x) dx \\
+ 2\gamma\theta' \int_{\infty}^{\infty} \left( G(x + s^{n(\tau)}(\theta')) - G(x) \right) h^\tau(x) dx - c(s^{n(\tau)}(\theta')),
\]

which by Expression (7), is strictly smaller than

\[
\int_{-\infty}^{\infty} \left( c(s^{n(\tau)}(\theta')) \right) h^\tau(x) dx + \int_{\infty}^{\infty} \left( c(s^{n(\tau)}(\theta')) \right) h^\tau(x) dx - c(s^{n(\tau)}(\theta')) \\
< c(s^{n(\tau)}(\theta')) - \varepsilon - c(s^{n(\tau)}(\theta')) < -\varepsilon,
\]

so playing \( a_i = 0 \) is strictly better, and hence \( s^{n(\tau)} \) is not an equilibrium. We have reached a contradiction. Thus, there cannot exist \( \theta' \in (-1, 1) \) such that \( \lim_{n \to +\infty} s^n(\theta') \neq 0 \), and it must be that \( \lim_{n \to +\infty} s^n(\theta) = 0 \) for each \( \theta \in (-1, 1) \).

We next establish an intermediate lemma (Lemma 6), which we use to prove that the ratio of marginal costs of two agents converges to their ratio of types (Lemma 7).

**Lemma 6** For any tuple \((F, \gamma, \{\bar{w}_n\}_{n=1}^{\infty}, p, c, G) \in \mathcal{F} \times \mathbb{R}_{++}^{\infty} \times \mathbb{R}_{++}^{\infty} \times \mathbb{R}_{++} \times C \times \mathcal{G} \), for any sequence \( \{s^n\}_{n=1}^{\infty} \) such that \( s^n \in E^{(n,F,\gamma,p,c,G)} \) for each \( n \in \mathbb{N}\backslash\{1\} \), for any \( n \in \mathbb{N}\backslash\{1\} \), and for
each \( \theta \in [-1, 0) \cup (0, 1] \), there exists \( z^\theta : [-\alpha(z^\theta), \alpha(z^\theta)] \rightarrow (s^n(\theta), 0) \cup (0, s^n(\theta)) \) such that \( \text{sgn}(z^\theta(x)) = \text{sgn}(\theta) \) for any \( x \in [-\alpha(z^\theta), \alpha(z^\theta)] \), and

\[
c'(s^n(\theta)) = 2\gamma \theta \left( \int_{-(n-1)^{-1}(z^\theta)}^{(n-1)^{-1}(z^\theta)} g(x) h^n(x) dx + s^n(\theta) \int_{-(n-1)^{-1}(z^\theta)}^{(n-1)^{-1}(z^\theta)} g'(x + z^\theta(x)) h^n(x) dx \right).
\]

(9)

**Proof.** For any given \( n \in \mathbb{N}\setminus\{1\} \), only a compact subset of the domain of \( G \), namely \( [-nc^{-1}(2\gamma), nc^{-1}(2\gamma)] \) is relevant, since \( ns^n(\theta) \in [-nc^{-1}(2\gamma), nc^{-1}(2\gamma)] \) for any \( \theta \). And \( G \) is twice continuously differentiable. Note that by the First Order Condition (3), for each \( \theta \in [-1, 1] \),

\[
c'(s^n(\theta)) = 2\gamma \theta \int_{-(n-1)^{-1}(z^\theta)}^{(n-1)^{-1}(z^\theta)} g(x + s^n(\theta)) h^n(x) dx.
\]

We want to show that for any \( x \in [-2nc^{-1}(2\gamma), nc^{-1}(2\gamma)] \), and any \( \theta \in (0, 1) \), there exists a \( z^\theta(x) \in (0, s^n(\theta)) \) such that

\[
g(x + s^n(\theta)) = g(x) + s^n(\theta)g'(x + z^\theta(x)).
\]

(10)

For each \( x \in [-2nc^{-1}(2\gamma), nc^{-1}(2\gamma)] \), define \( y_{\min}(x) \equiv \arg \min_{y \in [x, x + s^n(\theta)]} g'(y) \) and \( y_{\max}(x) \equiv \arg \max_{y \in [x, x + s^n(\theta)]} g'(y) \). Then note

\[
(s^n(\theta))g'(y_{\min}(x)) \leq g(x + s^n(\theta)) - g(x) \leq (s^n(\theta))g'(y_{\max}(x))
\]

Since \( g \) is continuous, by the Intermediate Value Theorem, there exists some value \( y(x) \in [x, x + s^n(\theta)] \) such that

\[
(s^n(\theta))g'(y(x)) = g(x + s^n(\theta)) - g(x).
\]

Then, define \( z^\theta(x) \equiv y(x) - x \) and we obtain Equality (10). An analogous argument, in this instance with \( y(x) \in [x + s^n(\theta), x] \), establishes that for any \( \theta \in [-1, 0) \), there exists a \( z^\theta(x) \in (s^n(\theta), 0) \) such that Equality (10) holds. \( \blacksquare \)

The next two lemmas use Lemma 6 to establish that the ratio of marginal costs of two
agents converges to their ratio of types (Lemma 7), and that the marginal effect of acquiring votes on the outcome converges to zero (Lemma 8).

**Lemma 7** For any \((F, \gamma, p, c, G) \in \mathcal{F} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times C \times \mathcal{G},\) for any sequence of equilibria \(\{s^n\}_{n=2}^{\infty},\) and for any \((\theta_1, \theta_2) \in (-1, 1)^2,\)

\[
\lim_{n \to \infty} \frac{c'(s^n(\theta_1))}{c'(s^n(\theta_2))} = \frac{\theta_1}{\theta_2}.
\]

**Proof.** For any \((F, \gamma, p) \in \mathcal{F} \times \mathbb{R}_{++} \times \mathbb{R}_{++},\) let \(\{s^n\}_{n=2}^{\infty}\) be such that \(s^n \in E^{(n,F,\gamma,p,c,G)}\) for each \(n \in \mathbb{N}\setminus\{1\}.

From Lemma 6, for each \(\theta \in [-1, 1],\)

\[
c'(s^n(\theta)) = 2\gamma\theta \left( \int_{-(n-1)c^{-1}(2\gamma)}^{(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx + s^n(\theta) \int_{-(n-1)c^{-1}(2\gamma)}^{(n-1)c^{-1}(2\gamma)} g'(x + z\theta(x))h^n(x)dx \right).
\]

Notice that since \(g\) is strictly positive and continuous, and \(g'\) is continuous, for any \(x, y \in \mathbb{R},\) \(g'(y)/g(x)\) is continuous, and it is bounded over any closed interval of \(\mathbb{R}.\) Further, by Condition (iii) of the definition of \(\mathcal{G},\) \(\exists \varepsilon \in \mathbb{R}_{++}\) such that for any \(\varepsilon \in (0, \varepsilon),\)

\[
\lim_{x \to -\infty} \frac{g'(x + \varepsilon)}{g(x)} \in \mathbb{R} \quad \text{and} \quad \lim_{x \to \infty} \frac{g'(x + \varepsilon)}{g(x)} \in \mathbb{R}.
\] (11)

Therefore, there exists \(\kappa \in \mathbb{R}_{++}\) such that \(\frac{g'(x + \varepsilon)}{g(x)} \in [-\kappa, \kappa],\) for any \(\varepsilon \in (0, \varepsilon)\) and for any \(x \in \mathbb{R}.\) Equivalently,

\[
-\kappa g(x) \leq g'(x + \varepsilon) \leq \kappa g(x) \quad \forall \varepsilon \in (0, \varepsilon), \quad \forall x \in \mathbb{R}.
\] (12)

Then, since \(s^n(\theta) \to 0\) for each \(\theta \in (-1, 1)\) and for any equilibrium strategy \(s^n\) (Lemma 5), it follows from Expression (12) that there exists \(\bar{n} \in \mathbb{N}\) such that for any \(n \in \mathbb{N}\) such that \(n > \bar{n},\) for each \(x \in [-(n-1)c^{-1}(2\gamma), (n-1)c^{-1}(2\gamma)],\) for any \(\theta \in [-1, 0) \cup (0, 1],\) and for any
equilibrium strategy $s^n$, we have:

$$-\kappa g(x) \leq g'(x + z^\theta(x)) \leq \kappa g(x),$$

where $z^\theta(x) \in (0, s^n(\theta))$ satisfies Equation (9). Then,

$$g(x) - s^n(\theta)\kappa g(x) \leq g(x) + s^n(\theta)g'(x + z^\theta(x)) \leq g(x) + s^n(\theta)\kappa g(x)$$

$$[1 - s^n(\theta)\kappa]g(x)\theta h^n(x) \leq (g(x) + s^n(\theta)g'(x + z^\theta(x)))\theta h^n(x) \leq (1 + s^n(\theta)\kappa)g(x)\theta h^n(x).$$

Once again since $s^n(\theta) \to 0$ for each $\theta \in (-1, 1)$ (Lemma 5), there exists $\tilde{n}$ such that $1 - s^n(\theta)\kappa > 0$ for every $n > \tilde{n}$.

That is, we can integrate $x$ over $[-(n - 1)c^{-1}(2\gamma), (n - 1)c^{-1}(2\gamma)]$ and obtain:

$$2\gamma[1 - s^n(\theta)\kappa]\theta \int_{-(n-1)c^{-1}(2\gamma)}^{(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx$$

$$\leq 2\gamma\theta \int_{-(n-1)c^{-1}(2\gamma)}^{(n-1)c^{-1}(2\gamma)} (g(x) + s^n(\theta)g'(x + z^\theta(x)))h^n(x)dx$$

$$\leq 2\gamma(1 + s^n(\theta)\kappa)\theta \int_{-(n-1)c^{-1}(2\gamma)}^{(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx,$$

and hence, substituting Equality (9), for any $\theta \in [-1, 0) \cup (0, 1],

$$c'(s^n(\theta)) \in \left[2\gamma(1 - s^n(\theta)\kappa)\theta \int_{-(n-1)c^{-1}(2\gamma)}^{(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx, 2\gamma(1 + s^n(\theta)\kappa)\theta \int_{-(n-1)c^{-1}(2\gamma)}^{(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx \right].$$

Then, for any $\theta, \theta' \in [-1, 0) \cup (0, 1],

$$\frac{c'(s^n(\theta))}{c'(s^n(\theta'))} \leq \left(\begin{array}{c} 2\gamma(1 - s^n(\theta)\kappa)\theta \int_{-(n-1)c^{-1}(2\gamma)}^{(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx \\ 2\gamma(1 + s^n(\theta')\kappa)\theta' \int_{-(n-1)c^{-1}(2\gamma)}^{(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx \\ 2\gamma(1 + s^n(\theta)\kappa)\theta \int_{-(n-1)c^{-1}(2\gamma)}^{(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx \\ 2\gamma(1 + s^n(\theta')\kappa)\theta' \int_{-(n-1)c^{-1}(2\gamma)}^{(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx \end{array}\right)$$

30
Note that because \( \lim_{n \to \infty} s^n(\tilde{\theta}) = 0 \) for any \( \tilde{\theta} \in (-1, 0) \cup (0, 1) \) (Lemma 5) and \( s^n(0) = 0 \) for any \( n \in \mathbb{N} \), both limit points of the interval converge to \( \frac{\theta}{\theta'} \) as \( n \) increases to infinity. Hence,

\[
\lim_{n \to \infty} \frac{c'(s^n(\theta))}{c'(s^n(\theta'))} = \frac{\theta}{\theta'}.
\]

Lemma 8. For any \((F, \gamma, p, c, G) \in \mathcal{F} \times \mathbb{R}^+ \times \mathbb{R}^+ \times C \times G\), and for any sequence of equilibria \(\{s^n\}_{n=2}^\infty\),

\[
\lim_{n \to \infty} \int_{-(n-1)c^{-1}(2\gamma)}^{-(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx = 0.
\]

Proof. By definition of \(C\), there exists a \( \lambda \in \mathbb{R}^+\) such that \( c' \) is strictly increasing in \((0, \lambda]\). Therefore, \( c' \) is invertible over \((0, \lambda]\). Let \((c')^{-1}\) denote the inverse of \( c' \) over \((0, \lambda]\). Then, for any \( \theta \in (-1, 1) \), from Expression (13) in the proof of Lemma 7,

\[
s^n(\theta) \in \begin{pmatrix}
(c')^{-1} \left(2\gamma(1 - s^n(\theta)\kappa) \int_{-(n-1)c^{-1}(2\gamma)}^{-(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx\right), \\
(c')^{-1} \left(2\gamma(1 + s^n(\theta)\kappa) \int_{-(n-1)c^{-1}(2\gamma)}^{-(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx\right)
\end{pmatrix}
\]

and, since \( \lim_{n \to \infty} s^n(\theta) = 0 \) for any \( \theta \in (-1, 1) \) (Lemma 5), it must be that

\[
\lim_{n \to \infty} (c')^{-1} \left(2\gamma(1 - s^n(\theta)\kappa) \int_{-(n-1)c^{-1}(2\gamma)}^{-(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx\right) = 0,
\]

which implies

\[
\lim_{n \to \infty} \left(2\gamma(1 - s^n(\theta)\kappa) \int_{-(n-1)c^{-1}(2\gamma)}^{-(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx\right) = 0,
\]

which, for any \( \theta \in (-1, 0) \cup (0, 1) \), implies

\[
\lim_{n \to \infty} \int_{-(n-1)c^{-1}(2\gamma)}^{-(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx = 0.
\]
We next define an auxiliary function and prove a lemma related to it. Define $J : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+$ by

$$J(x, y) = \begin{cases} \frac{y c''(y)}{c'(y)} & \text{if } x = y \\ \frac{\ln c'(x) - \ln c'(y)}{\ln x - \ln y} & \text{otherwise} \end{cases}.$$  

**Lemma 9** Let $\{x_n\}_{n=1}^{\infty} \in \mathbb{R}^\infty_+$ and $\{y_n\}_{n=1}^{\infty} \in \mathbb{R}^\infty_+$ be two converging sequences with $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0$ and define $z \equiv \lim_{x \to 0} \frac{xc'(x)}{c(x)}$. Then $\lim_{n \to \infty} J(x_n, y_n) = z - 1$.

**Proof.** Note that for any $y \in \mathbb{R}_+$,

$$\lim_{x \to 0} J(x, y) = \frac{\ln c'(0) - \ln c'(y)}{\ln 0 - \ln y} = \frac{-\infty}{-\infty},$$

applying L’Hospital rule,

$$\lim_{x \to 0} J(x, y) = \lim_{x \to 0} \frac{\frac{c'(x)}{c(x)} - \frac{xc''(x)}{c'(x)}}{\frac{1}{x}} = \lim_{x \to 0} \frac{xc''(x)}{c'(x)}.$$

Notice that $z \equiv \lim_{x \to 0} \frac{xc'(x)}{c(x)} = \frac{0}{0}$, so applying L’Hospital rule,

$$z = \lim_{x \to 0} \frac{c'(x) + xc''(x)}{c'(x)} = 1 + \lim_{x \to 0} \frac{xc''(x)}{c'(x)},$$

$$z - 1 = \lim_{x \to 0} \frac{xc''(x)}{c'(x)},$$

so $\lim_{x \to 0} J(x, y) = z - 1$. Note as well that, using L’Hospital rule

$$\lim_{\varepsilon \to 0} J(x, x + \varepsilon) = \frac{\frac{c''(x)}{c'(x)} - \frac{1}{x}}{\frac{1}{x}} = \frac{xc''(x)}{c'(x)}$$

so $J$ is continuous.
Define the function \( v : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) by

\[
v(x) = \begin{cases} 
z - 1 & \text{if } x = 0 \\
\frac{xe''(x)}{c'(x)} & \text{if } x \in \mathbb{R}_+^+ \end{cases}.
\]

Since

\[
\lim_{x \to 0} v(x) = \lim_{x \to 0} \frac{xc''(x)}{c'(x)}
\]

and hence by Equality (15), \( \lim_{x \to 0} v(x) = z - 1 \) and hence \( v \) is continuous.

Define the correspondence \( x^+ : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) by \( x^+(w) = \arg \max_{x \in [0,w]} v(x) \) for each \( w \in \mathbb{R}_+ \), and the correspondence \( x^- : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) by \( x^-(w) = \arg \min_{x \in [0,w]} v(x) \) for each \( w \in \mathbb{R}_+ \), and define the function \( v^+ : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) by \( v^+(w) = \max_{x \in [0,w]} v(x) \) for each \( w \in \mathbb{R}_+ \) and the function \( v^- : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) by \( v^-(w) = \min_{x \in [0,w]} v(x) \) for each \( w \in \mathbb{R}_+ \). Since \( v \) is continuous, \( x^+(w) \) and \( x^-(w) \) are non-empty for each \( w \in \mathbb{R}_+ \), \( x^+ \) and \( x^- \) are upper hemi continuous, and \( v^+ \) and \( v^- \) are continuous (Berge’s maximum theorem). Further, note that \( v^+ \) is non-decreasing and \( v^- \) is non-increasing.

Construct two sequences \( \{x_t\}_{t=1}^{\infty} \in \mathbb{R}_+\infty \) and \( \{y_t\}_{t=1}^{\infty} \in \mathbb{R}_+\infty \) such that \( \lim_{t \to \infty} x_t = \lim_{t \to \infty} y_t = 0 \). Then

\[
\lim_{t \to 0} \frac{x_tc''(x_t)}{c'(x_t)} = \lim_{t \to 0} \frac{ytc''(y_t)}{c'(y_t)} = z - 1.
\]

Note that for any \( y \in \mathbb{R}_+^+ \), and for any \( x \in (0, y) \), \( J \) is differentiable and

\[
\frac{\partial J}{\partial x}(x, y) = \frac{c''(x)(\ln x - \ln y) - (\ln c'(x) - \ln(c'(y)) \frac{1}{x}}{(\ln x - \ln y)^2} = \frac{xe''(x)(\ln x - \ln y) - c'(x)(\ln c'(x) - \ln(c'(y)))}{xc'(x)(\ln x - \ln y)^2}.
\]
Hence \( \frac{\partial J}{\partial x}(x, y) = 0 \) if and only if
\[
\frac{xc''(x) \ln x - y}{c'(x)} = \frac{c'(x) (\ln c'(x) - \ln(c'(y)))}{\ln x - \ln y},
\]
that is, \( \frac{\partial J}{\partial x}(x, y) = 0 \) if and only if \( J(x, y) = \frac{xc''(x)}{c'(x)} \).

Since \( x \in \arg \max_{y \in (0, y)} J(x, y) \) implies \( \frac{\partial J}{\partial x}(x, y) = 0 \), it follows that for any \( y \in \mathbb{R}_{++} \) and any \( x \in \arg \max_{y \in (0, y)} J(x, y) \), \( J(x, y) = v(x) \), so \( J(x, y) \leq v^+(x) \). Since \( v^+ \) is non-decreasing, it follows that \( \max_{x \in (0, y)} J(x, y) \leq v^+(y) \). If \( \arg \max_{y \in (0, y)} J(x, y) = \emptyset \), then
\[
\sup_{y \in (0, y)} \left\{ \lim_{x \to 0} J(x, y) \right\} = \{ z - 1, v(y) \} \leq v^+(y).
\]
So \( \sup_{y \in (0, y)} J(x, y) \leq v^+(y) \) for any \( y \in \mathbb{R}_{++} \). Similarly, it can be shown that \( \sup_{y \in (0, x)} J(x, y) \leq v^+(x) \) for any \( x \in \mathbb{R}_{++} \).

Moreover, since \( x \in \arg \min_{x \in (0, y)} J(x, y) \) implies \( \frac{\partial J}{\partial x}(x, y) = 0 \), it follows that for any \( y \in \mathbb{R}_{++} \) and any \( x \in \arg \min_{x \in (0, y)} J(x, y) \), \( J(x, y) = v(x) \), so \( J(x, y) \geq v^-(x) \). Since \( v^- \) is non-decreasing, it follows that \( \max_{x \in (0, y)} J(x, y) \geq v^-(y) \). If \( \arg \min_{x \in (0, y)} J(x, y) = \emptyset \), then
\[
\inf_{x \in (0, y)} \left\{ \lim_{x \to 0} J(x, y) \right\} = \{ z - 1, v(y) \} \geq v^-(y).
\]
So \( \inf_{x \in (0, y)} J(x, y) \geq v^-(y) \) for any \( y \in \mathbb{R}_{++} \). Similarly, it can be shown that \( \inf_{y \in (0, x)} J(x, y) \geq v^-(y) \) for any \( x \in \mathbb{R}_{++} \).

From all the above it follows that for any \( t \in \mathbb{N} \), \( J(x_t, y_t) \in [v^-(w_t), v^+(w_t)] \), where \( w_t = \max\{x_t, y_t\} \). Notice that \( \lim_{t \to \infty} w_t = 0 \), and thus \( \lim_{t \to 0} v^-(w_t) = z - 1 \) and \( \lim_{t \to 0} v^+(w_t) = z - 1 \), and hence \( \lim_{n \to -\infty} J(x_n, y_n) = z - 1 \).

We next establish the key intermediary result: equilibrium actions are asymptotically linear in \( (\theta)^p \).

**Lemma 10** Let \( \{s^n\}_{n=1}^\infty \) denote a sequence of equilibria given \( (F, \gamma, \{w_n\}_{n=1}^\infty, p, c, G) \in \mathcal{F} \times \mathbb{R}_{++} \times \mathbb{R}^\infty_{++} \times \mathbb{R}_{++} \times C \times \mathcal{G} \) and let \( z \equiv \lim_{x \to 0} \frac{xc'(x)}{c(x)} \). Then for any \( \theta_1, \theta_2 \in (-1, 1) \),
\[
\lim_{n \to +\infty} \frac{s^n(\theta_1)}{s^n(\theta_2)} = sgn \left( \frac{\theta_1}{\theta_2} \right) \left| \frac{\theta_1}{\theta_2} \right|^{\frac{1}{z-1}}.
\]
Proof. For any \((\theta_1, \theta_2) \in (-1, 0)^2 \cup (0, 1)^2\), by Lemma 7, \(\lim_{n \to \infty} \frac{c'(s^n(\theta_1))}{c'(s^n(\theta_2))} = \frac{\theta_1}{\theta_2}\), and taking logarithms on both sides,

\[
\lim_{n \to \infty} \left( \ln c'(s^n(\theta_1)) - \ln c'(s^n(\theta_2)) \right) = \ln \left( \frac{\theta_1}{\theta_2} \right). \tag{17}
\]

By Lemma 9, for any \(\{x_n\}_{n=1}^{\infty} \in \mathbb{R}^{\infty}_{++}\) with \(\lim_{n \to \infty} x_n = 0\) and \(\{y_n\}_{n=1}^{\infty} \in \mathbb{R}^{\infty}_{++}\) with \(\lim_{n \to \infty} y_n = 0\),

\[
\lim_{n \to \infty} \frac{\ln c'(x_n) - \ln c'(y_n)}{\ln \frac{x_n}{y_n}} = z - 1,
\]

thus, in particular,

\[
\lim_{n \to \infty} \frac{\ln c'(s^n(\theta_1)) - \ln c'(s^n(\theta_2))}{\ln \frac{s^n(\theta_1)}{s^n(\theta_2)}} = z - 1,
\]

and thus substituting the left hand side according to Equality 17, we obtain

\[
\ln \frac{\theta_1}{\theta_2} = \lim_{n \to \infty} \ln \left( \frac{s^n(\theta_1)}{s^n(\theta_2)} \right)^{z-1},
\]

\[
\lim_{n \to \infty} \frac{s^n(\theta_1)}{s^n(\theta_2)} = \left( \frac{\theta_1}{\theta_2} \right)^{\frac{1}{z-1}}. \tag{18}
\]

For any \(\theta_1 \in (-1, 0)\) and \(\theta_2 \in (0, 1)\), by neutrality of \(s^n\), \(s^n(\theta_1) = \text{sgn}(\theta_1)s^n(|\theta_1|)\), so

\[
\lim_{n \to \infty} \frac{s^n(\theta_1)}{s^n(\theta_2)} = \lim_{n \to \infty} \frac{\text{sgn}(\theta_1)s^n(|\theta_1|)}{s^n(\theta_2)} = \text{sgn}(\theta_1) \left( \frac{|\theta_1|}{\theta_2} \right)^{\frac{1}{z-1}} = \text{sgn} \left( \frac{\theta_1}{\theta_2} \right) \left| \frac{\theta_1}{\theta_2} \right|^{\frac{1}{z-1}},
\]

with an identical result up to relabeling if \(\theta_2 \in (-1, 0)\) and \(\theta_1 \in (0, 1)\). 

We can now prove Theorem 3.

Proof of Theorem 3. Part I. Sufficient conditions. We first prove that if there exists \(\rho \in (0, \infty)\) such that \(SC\) and \(SC'_\rho\) converge to each other almost everywhere, then any
vote-buying mechanism $c \in C$ such that $\lim_{x \to 0^+} \frac{xc'(x)}{c(x)} = \frac{1+\rho}{\rho}$ asymptotically implements $SC$.

By Lemma 8,

$$\lim_{n \to \infty} \int_{-(n-1)c^{-1}(2\gamma)}^{(n-1)c^{-1}(2\gamma)} g(x)h^n(x)dx = 0. \quad (19)$$

Since $g(x) > 0$ for any $x \in \mathbb{R}$, Expression (19) implies that for any $\bar{x} \in \mathbb{R}_+$,

$$\lim_{n \to \infty} \int_{\bar{x}}^{\bar{x}} g(x)h^n(x)dx = 0. \quad (20)$$

Since $g$ is continuous, it attains a minimum in $[-\bar{x}, \bar{x}]$, and this minimum is strictly positive. Since $h^n(x) \in \mathbb{R}_+$ for any $x \in \mathbb{R}$ and for any $n \in \mathbb{N} \setminus \{1\}$, Expression (20) implies that

$$\lim_{n \to \infty} \int_{-\bar{x}}^{\bar{x}} h^n(x)dx = 0, \quad (21)$$

which implies that for any $\bar{x}$ and any $x \in [-\bar{x}, \bar{x}]$,

$$\lim_{n \to \infty} (H(\bar{x}) - H(-\bar{x})) = 0. \quad (22)$$

For each $n \in \mathbb{N} \setminus \{1\}$, let $(\bar{\theta}_1, ..., \bar{\theta}_k, ..., \bar{\theta}_n)$ be $n$ independent, identically distributed random variables drawn from $F$. Selecting $\bar{x}$ such that $G(\bar{x}) > 1 - \frac{\xi}{2}$, it follows from Expression (22) and from $\lim_{n \to \infty} s^n(\theta) = 0$ for each $\theta \in \mathbb{N}$ (Lemma 5) that

$$\lim_{n \to \infty} \Pr \left[ G \left( \sum_{k \in \mathbb{N}^n} s^n(\bar{\theta}_k) \right) \in \left( \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2} \right) \right] = 0. \quad (23)$$

We want to show that

$$\lim_{n \to \infty} \frac{2^n}{L(\Theta_A(SC^n))} \int_{\Theta_n \in \Theta_A(SC^n)} \left( \prod_{i=1}^{n} f(\theta_i) \right) G \left( \sum_{i \in \mathbb{N}^n} s^n(\theta_i) \right) d\theta_N > 1 - \varepsilon. \quad (24)$$

(an analogous argument applies to $\theta \in \Theta_B(SC^n)$). Assume that there exists $\rho \in \mathbb{R}_+$ such
that $SC_n$ and $SC^*_n$ converge to each other. Then, for any $(\gamma, p) \in \mathbb{R}_+^2$,

$$\lim_{n \to \infty} \frac{L(\Theta_J(SC^n) \cap (\Theta_J(SC^*_n))^c)}{2^n} = 0$$

for each $J \in \{A, B\}$,

where

$$\Theta_A(SC^n) \equiv \left\{ (\gamma, p, \theta_N^n) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times [-1, 1]^n : \sum_{i \in N^n} sgn(\theta_i)|\theta_i|^\rho > 0 \right\}$$

and

$$\Theta_B(SC^n) \equiv \left\{ (\gamma, p, \theta_N^n) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times [-1, 1]^n : \sum_{i \in N^n} sgn(\theta_i)|\theta_i|^\rho < 0 \right\}.$$

Thus, in order for Inequality (24) to hold, it suffices that

$$\lim_{n \to \infty} \frac{2^n}{L(\Theta_A(SC^n))} \int_{(\gamma, p, \theta_N^n) \in \Theta_A(SC^*_n)} \left( \prod_{i=1}^n f(\theta_i) \right) G \left( \sum_{i \in N^n} s^n(\theta_i) \right) d\theta_N > 1 - \varepsilon.$$

Note $\lim_{x \to 0} \frac{x^\rho}{\rho + 1} = \frac{1}{\rho + 1}$ so $\rho = \frac{1}{\lim_{x \to 0} \frac{x^\rho}{\rho + 1} - 1}$. For each $n \in \mathbb{N} \setminus \{1\}$, by Lemma 10,

$$\lim_{n \to \infty} \frac{s^n(\theta)}{n} = \frac{sgn(\theta)(2|\theta|)\lim_{x \to 0} \frac{1}{\frac{x^\rho}{\rho + 1} - 1}}{n} = \frac{sgn(\theta)2^\rho|\theta|^\rho}{n}$$

for each $\theta \in (-1, 1)$. (25)

For each $n \in \mathbb{N} \setminus \{1\}$, define the random variable $\rho^n(\bar{\theta}) \equiv \frac{s^n(\bar{\theta})}{s^n(\frac{1}{2})} - sgn(\bar{\theta})2^\rho|\bar{\theta}|^\rho$. By Equality (25), for any $\delta \in \mathbb{R}_+$, there exists $\hat{n}_\delta \in \mathbb{N}$ such that for any $n > \hat{n}_\delta$, $\rho^n(\theta) \in (-\delta, \delta)$ for any $\theta \in (0, 1)$, $\rho^n(\theta) = -\rho^n(-\theta)$ for any $\theta \in (-1, 0)$ and $\rho^n(0) = 0$. So, for any $n > \hat{n}_\delta$, $Var(\rho^n(\bar{\theta})) \leq \delta^2$. We can then construct a decreasing sequence $\{\delta_t\}_{t=1}^\infty$ such that $\delta_t \xrightarrow{t \to \infty} 0$, and obtain

$$\lim_{n \to \infty} Var(\rho^n(\bar{\theta})) = 0.$$ (26)

For each $n \in \mathbb{N} \setminus \{1\}$, and for each $k \in \{1, \ldots, n\}$, define the random variable $\rho_k^n(\bar{\theta}) \equiv \frac{s^n(\bar{\theta})}{s^n(\frac{1}{2})} - sgn(\bar{\theta})2^\rho|\bar{\theta}|^\rho$. These are $n$ independent, identically distributed random variables. Then note
that
\[
\operatorname{Var} \left( \frac{\sum_{k=1}^{n} \rho_k^n(\bar{\theta})}{\sqrt{n}} \right) = \operatorname{Var}(\rho^n(\bar{\theta})),
\] (27)
so by equalities (26) and (27),
\[
\lim_{n \to \infty} \operatorname{Var} \left( \frac{\sum_{k=1}^{n} \rho_k^n(\bar{\theta})}{\sqrt{n}} \right) = 0;
\]
that is, as \( n \to \infty \) the realization of \( \frac{\sum_{k=1}^{n} \rho_k^n(\bar{\theta})}{\sqrt{n}} \) becomes arbitrarily close to zero with probability converging to one, so the cumulative distribution of \( \frac{\sum_{k=1}^{n} \rho_k^n(\bar{\theta})}{\sqrt{n}} \) converges to a step function that is zero below zero, and one above zero. Similarly, \( \operatorname{Var} \left( \frac{\sum_{k=1}^{n} \text{sgn}(\theta_k)2^{|\theta_k|^\rho}}{\sqrt{n}} \right) = \operatorname{Var}(\text{sgn}(\bar{\theta})2^{|\bar{\theta}|^\rho}) > 0 \), so the distribution of \( \frac{\sum_{k=1}^{n} \text{sgn}(\theta_k)2^{|\theta_k|^\rho}}{\sqrt{n}} \) converges to a normal distribution with mean zero and strictly positive variance equal to \( \operatorname{Var}(\text{sgn}(\bar{\theta})2^{|\bar{\theta}|^\rho}) \). Hence,
\[
\lim_{n \to \infty} \Pr \left[ \text{sgn} \left( \frac{\sum_{i \in N^n} s^n(\theta_i)}{\sqrt{n}} \right) \neq \text{sgn} \left( \frac{\sum_{i \in N^n} \text{sgn}(\theta_i)2^{|\theta_i|^\rho}}{\sqrt{n}} \right) \right] = 0,
\]
or equivalently, since \( s^n(\frac{1}{2}) > 0 \) for each \( n \in \mathbb{N} \setminus \{1\} \),
\[
\lim_{n \to \infty} \Pr \left[ \text{sgn} \left( \sum_{i \in N^n} s^n(\theta_i) \right) \neq \text{sgn} \left( \sum_{i \in N^n} \text{sgn}(\theta_i)|\theta_i|^\rho \right) \right] = 0,
\] (28)
and Equality (23) together with Equality (28) implies
\[
\lim_{n \to \infty} \Pr \left[ G \left( \sum_{i \in N^n} s^n(\theta_i) \right) > 1 - \frac{\varepsilon}{2} \left| \text{sgn} \left( \sum_{i \in N^n} \text{sgn}(\theta_i)|\theta_i|^\rho \right) > 0 \right. \right] = 1.
\] (29)
Thus, subject to \((\gamma, p, \theta_{N^n}) \in \Theta_A(SC^n_{\rho(c)})\), with probability converging to one in \(n\), \(\sum_{i \in N^n} s^n(\theta_i)\) is strictly positive (Expression (28)), and subject to \(\sum_{i \in N^n} s^n(\theta_i)\) being strictly positive, its magnitude is sufficiently large so that \(G\left(\sum_{i \in N^n} s^n(\theta_i)\right) > 1 - \frac{\varepsilon}{2}\) (Expression (29)). Overall, subject to \((\gamma, p, \theta_{N^n}) \in \Theta_A(SC^n_{\rho}),\) if \(n\) is sufficiently large, \(G\left(\sum_{i \in N^n} s^n(\theta_i)\right) > 1 - \varepsilon\) as desired.

Hence, if there exists \(\rho \in (0, \infty)\) such that \(SC\) and \(SC_{\rho}\) converge to each other almost everywhere, then any vote-buying mechanism \(c \in C\) such that \(\lim_{x \to 0^+} \frac{xc'(x)}{c(x)} = \frac{1+\rho}{\rho}\) asymptotically implements \(SC\).

**Part II. Necessary conditions.**

Next we show that if for any \(\rho \in \mathbb{R}_{++}, SC\) and \(SC_{\rho}\) do not converge to each other almost everywhere, then for any \(c \in C\), vote-buying mechanism \(c\) does not asymptotically implement \(SC\), hence \(SC\) is not asymptotically implementable by any vote mechanism in \(C\).

Assume that for any \(\rho \in \mathbb{R}_{++}, SC\) and \(SC_{\rho}\) do not converge to each other almost everywhere. Let \(c \in C\) be an arbitrary vote buying mechanism. We want to show that \(c\) does not asymptotically implement \(SC\).

As established in Part I, \(c \in C\) asymptotically implements \(SC\) \(\lim_{x \to 0^+} \frac{xc'(x)}{c(x)} = \frac{1+\rho}{\rho}\) \(\in \mathbb{R}_{++}\) and denote \(\lim_{x \to 0^+} \frac{xc'(x)}{c(x)} = 1\) by \(\rho(c)\). Then, for any \((F, \{\tilde{w}_n\}_{n=1}^{\infty}, \gamma, p, G) \in \mathcal{F} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathcal{G}\), for any \(\varepsilon \in (0, 1)\), and for any sequence \(\{s^n\}_{n=2}^{\infty}\) such that \(s^n \in \mathcal{E}^{(n, F, \gamma, p, c, G)}\) for each \(n \in \mathbb{N}\backslash\{1\}\), there exists \(n_{\varepsilon, \gamma, F, p, G} \in \mathbb{N}\) such that for any \(n > n_{\varepsilon, \gamma, F, p, G}\),

\[
\frac{2^n}{L(\Theta_A(SC^n_{\rho(c)}))} \int_{(\gamma, p, \theta_{N^n}) \in \Theta_A(SC^n_{\rho(c)})} \left(\prod_{i=1}^{n} f(\theta_i)\right) G\left(\sum_{i=1}^{n} s^n(\theta_i)\right) d\theta_{N^n} > 1 - \varepsilon \quad \text{and} \quad (30)
\]

\[
\frac{2^n}{L(\Theta_B(SC^n_{\rho(c)}))} \int_{(\gamma, p, \theta_{N^n}) \in \Theta_B(SC^n_{\rho(c)})} \left(\prod_{i=1}^{n} f(\theta_i)\right) G\left(\sum_{i=1}^{n} s^n(\theta_i)\right) d\theta_{N^n} < \varepsilon.
\]

In particular, assume \(F\) is a uniform distribution. Since for any \(\rho \in \mathbb{R}_{++}, SC\) and \(SC_{\rho}\) do not converge to each other almost everywhere and since \(\rho(c) \in \mathbb{R}_{++}\), in particular \(SC\) and
$SC_{p(\alpha)}$ do not converge to each other almost everywhere; that is, there exists $(\gamma, p) \in \mathbb{R}^2_{++}$ and $J \in \{A, B\}$ such that

$$\lim_{n \to \infty} \frac{L(\Theta_J(SC^n) \cap (\Theta_J(\{SC^n_{\rho(\alpha)}\})^c))}{2^n} \neq 0.$$  

Then there exists $\delta \in \mathbb{R}_{++}$, $J \in \{A, B\}$, $(\gamma, p) \in \mathbb{R}^2_{++}$ and a strictly increasing sequence $\{n(\tau)\}_{\tau=1}^\infty \in \mathbb{N}^\infty$ such that

$$\frac{L(\Theta_J(SC^{n(\tau)}) \cap (\Theta_J(SC^{n(\tau)}_{\rho(\alpha)})))^c}{2^{n(\tau)}} > \delta \text{ for each } \tau \in \mathbb{N}.$$  

Without loss of generality, assume

$$\frac{L(\Theta_A(SC^{n(\tau)}) \cap \Theta_A((SC^{n(\tau)}_{\rho(\alpha)})))^c}{2^{n(\tau)}} > \delta \text{ for each } \tau \in \mathbb{N}.$$  

Since $L(\{(\gamma, p, \theta_{N^n}) \in \mathbb{R}_{++} \times \mathbb{R}_{++} \times [-1, 1]^n : SC^{n}_{\rho(\alpha)}(\gamma, p, \theta_{N^n}) = \{A, B\}\}) = 0$,

$$L(\Theta_A(SC^{n(\tau)}) \cap (\Theta_A(SC^{n(\tau)}_{\rho(\alpha)})))^c) = L(\Theta_A(SC^{n(\tau)}) \cap \Theta_B(SC^{n(\tau)}_{\rho(\alpha)})),$$

so

$$L(\Theta_A(SC^{n(\tau)}) \cap \Theta_B(SC^{n(\tau)}_{\rho(\alpha)})) > \delta 2^{n(\tau)}.$$  

Since, by assumption, $F$ is uniform, $\prod_{i=1}^{n(\tau)} f(\theta_i) = \frac{1}{2^{n(\tau)}}$, and thus

$$\int_{(\gamma, p, \theta_{N^n(\tau)}) \in \Theta_A(SC^{n(\tau)} \cap \Theta_B(SC^{n(\tau)}_{\rho(\alpha)}))} \prod_{i=1}^{n(\tau)} f(\theta_i) d\theta_{N^n(\tau)} > \delta.$$  

For any $\varepsilon \in (0, \frac{\delta}{2})$, implementing $SC$ requires that for some $\hat{n}_{\varepsilon, \gamma, F, p, G} \in \mathbb{N}$, and for any
\[ n > \hat{n}_{\varepsilon, \gamma, F, p, G} \in \mathbb{N} \]
\[
\frac{2^n}{L(\Theta_A(SC^n))} \int_{(\gamma, p, \theta_N^n) \in \Theta_A(SC^n)} \left( \prod_{i=1}^{n} f(\theta_i) \right) G \left( \sum_{i=1}^{n} s^n(\theta_i) \right) d\theta_N^n > 1 - \varepsilon
\]
\[
\frac{1}{L(\Theta_A(SC^n))} \left( \int_{(\gamma, p, \theta_N^n) \in \Theta_A(SC^n) \cap \Theta_B(SC^n_{\rho(c)})} G \left( \sum_{i=1}^{n} s^n(\theta_i) \right) d\theta_N^n \right) > 1 - \varepsilon
\]
which implies
\[
\frac{1}{L(\Theta_A(SC^n))} \left( \int_{(\gamma, p, \theta_N^n) \in \Theta_A(SC^n) \cap \Theta_B(SC^n_{\rho(c)})} G \left( \sum_{i=1}^{n} s^n(\theta_i) \right) d\theta_N^n \right) > 1 - \varepsilon
\]
\[
1 - \delta + \frac{1}{L(\Theta_A(SC^n))} \int_{(\gamma, p, \theta_N^n) \in \Theta_A(SC^n) \cap \Theta_B(SC^n_{\rho(c)})} G \left( \sum_{i=1}^{n} s^n(\theta_i) \right) d\theta_N^n > 1 - \varepsilon
\]
so
\[
\frac{2^n}{L(\Theta_A(SC^n))} \int_{(\gamma, p, \theta_N^n) \in \Theta_A(SC^n) \cap \Theta_B(SC^n_{\rho(c)})} \left( \prod_{i=1}^{n} f(\theta_i) \right) G \left( \sum_{i=1}^{n} s^n(\theta_i) \right) d\theta_N^n > \varepsilon,
\]
and
\[
\frac{2^n}{L(\Theta_A(SC^n))} \int_{(\gamma, p, \theta_N^n) \in \Theta_B(SC^n_{\rho(c)})} \left( \prod_{i=1}^{n} f(\theta_i) \right) G \left( \sum_{i=1}^{n} s^n(\theta_i) \right) d\theta_N^n > \varepsilon,
\]
which cannot be satisfied, since \( c \) implements \( SC_{\rho(c)} \). Hence, \( c \) does not implement \( SC \), and since \( c \) was arbitrary, \( SC \) is not implementable. \[ \blacksquare \]
References


